Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 3, Exercise class

Exercise 1:

Consider the the initial value problem

$$y'''(t) - 2y''(t) - y'(t) + 2y(t) = 3\sin(t)\,, \qquad y(0) = 0,\, y'(0) = 1,\, y''(0) = \frac{3}{10}.$$

be given.

a) Which order has the differential equation ?

b) Is it an explicit differential equation ? If not provide an equivalent explicit differential equation.

c) Is the differential equation linear?

d) Is the differential equation homogeneous?

e) Rewrite the initial value problem into an equivalent initial value problem for a system of first order.

Solution:

a) The differential equation has oder three.

b) No. Explicit form:

$$y'''(t) = 2y''(t) + y'(t) - 2y(t) + 3\sin(t)\,.$$

c) Yes.

d) No.

e) With

$$\boldsymbol{y} := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, \qquad \boldsymbol{y}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix}$$

one obtains as an equivalent system

$$\boldsymbol{y}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3\sin(t) \end{pmatrix}, \qquad \boldsymbol{y}(0) := \begin{pmatrix} 0 \\ 1 \\ \frac{3}{10} \end{pmatrix}.$$

or

$$\boldsymbol{y}' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3\sin(t) \end{pmatrix}, \qquad \boldsymbol{y}(0) := \begin{pmatrix} 0 \\ 1 \\ \frac{3}{10} \end{pmatrix}.$$

Exercise 2:

Determine the solutions of the following initial value problems

a)
$$y'(t) = \frac{1+\cos(t)}{(y(t))^2} \qquad \text{for } t > 0, \qquad y(0) = 3.$$

b)
$$y - ty' = \frac{t^3}{y^2} \qquad \text{for } t > 1, \qquad y(1) = 2.$$

Hint: Substitute $u(t) := \frac{y(t)}{t}$.

Solution:

a) It is a separable differential equation. For $y \neq 0$ one computes
$\frac{dy}{dt} = \frac{1+\cos(t)}{y^2} \iff y^2 dy = (1+\cos(t))dt.$
From this
$\int y^2 dy = \int (1+\cos(t))dt \iff \frac{y^3}{3} = t + \sin(t) + C$

$\iff y(t) = \sqrt[3]{3t + 3\sin(t) + 3C}.$
$y(0) = \sqrt[3]{0 + 3\sin(0) + 3C} \stackrel{!}{=} 3 \iff C = 9\,.$
Thus $y(t) = \sqrt[3]{3t + 3\sin(t) + 27}\,.$

b) Solving with respect to y' yields
$$y' = \frac{y}{t} - \left(\frac{t}{y}\right)^2.$$
The substitution $u(t) = \frac{y(t)}{t}$ converts the differential equation $y' = \frac{y}{t} - \left(\frac{t}{y}\right)^2$ into a separable variable differential equation. Since
$y'(t) = (t \cdot u(t))' = u(t) + tu'(t)$
the differential equation requires
$u(t) + tu'(t) \stackrel{!}{=} u(t) - \left(u^{-1}(t)\right)^2 \stackrel{t\neq 0}{\iff} u'(t) \stackrel{!}{=} \frac{u(t) - u^{-2}(t) - u(t)}{t}\,.$
Hence one has to solve
$$u' = \frac{-u^{-2}}{t}.$$
This differential equation is separable. The solution can be computed as follows:
$$\int u^2 du = \int \frac{-dt}{t}$$
$$\frac{u^3}{3} = -\ln(t) + \tilde{C}$$
$$u = (-3\ln(t) + C)^{\frac{1}{3}}$$
$$y = t \cdot u = t\,(-3\ln(t) + C)^{\frac{1}{3}}$$

Initial condition:
$y(1) = 1 \cdot t\,(-3\ln(1) + C)^{\frac{1}{3}} \stackrel{!}{=} 2 \implies C = 8.$

Dates of classes: 13.11-17.11.2023.