Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I
for Students of Engineering Sciences

Sheet 2, Homework

Exercise 1:

We consider the differential equation $y' - 2y = 4t$.

a) Show that $y_p(t) = -2t - 1$ solves the differential equation.

b) Let $\tilde{y}$ be a second solution of the differential equation.. Show that $y := \tilde{y} - y_p$ solves the corresponding homogeneous differential equation $y' - 2y = 0$.

c) Compute solutions y_h of the corresponding homogeneous differential equation $y_h' - 2y_h = 0$, for example with the help of the formula on page 9 of the lecture.

d) From b) it follows that every solution of the inhomogeneous differential equation can be written as sum of a solution of the homogeneous differential equation and y_p . Calculate the solution of the initial value problem
$$y' - 2y = 4t, \qquad y(0) = 2.$$

Solution:

a) Inserting $y_p(t) = -2t - 1$ into the differential equation delivers:
$-2 - 2(-2t - 1) \;=\; -2 + 4t + 2 \;=\; 4t$.

b) It holds
$$y' - 2y = (\tilde{y}' - y_p') - 2(\tilde{y} - y_p) = \underbrace{(\tilde{y}' - 2\tilde{y})}_{4t} - \underbrace{(y_p' - 2y_p)}_{4t} \;=\; 0.$$

c) The solutions y_h of the corresponding homogeneous differential equation $y_h' - 2y_h = 0$ can either be guessed, determined according to the formula on page 5 of the lecture, or computed according to page 9 of the lecture.

Using the notation of the lecture we have $a(t) = 2$ thus

$A(t) = 2t + \hat{C}$ and from this $y_h(t) = e^{\hat{C}} e^{2t} = Ce^{2t}$.

d) From a) and b) we obtain $y(t) = -2t - 1 + Ce^{2t}$.

The initial condition requires

$y(0) = 0 - 1 + Ce^0 = C - 1 \overset{!}{=} 2 \Longrightarrow C = 3$.

Thus the solution of the initial value problem is $y(t) = -2t - 1 + 3e^{2t}$.

Exercise 2:

Analogously to the procedure on page 9 of the lecture, prove the representation given on page 11

$$u(t) = e^{A(t)}(B^*(t) + C), \qquad \text{with } A'(t) = a(t),\ (B^*(t))' = e^{-A(t)} \cdot b(t),\ C \in \mathbb{R} \qquad (*)$$

for any solution of the differential equation

$$u'(t) = a(t) \cdot u(t) + b(t).$$

Where the same assumptions of the lecture apply to a and b.

Hint:
Prove first, by a substitution into the differential equation , that a solution is given by $(*)$.

Conversely, assume that you have a solution u and, analogous to the procedure on page 9 of the lecture, show that $e^{-A(t)}u(t) - B^*(t)$ is constant.

Solution:

First of all, with (*) it holds

$$u'(t) = A'(t)e^{A(t)}(B^*(t) + C) + e^{A(t)}(B^*(t))' = a(t)\underbrace{e^{A(t)}(B^*(t) + C)}_{u(t)} + \underbrace{e^{A(t)}(e^{-A(t)} \cdot b(t))}_{b(t)} = a(t)u(t) + b(t).$$

On the other hand, let u be a solution of the differential equation . Then $u'(t) = a(t)u(t) + b(t)$ and thus

$$\Big(e^{-A(t)}u(t) - B^*(t)\Big)' = -a(t)e^{-A(t)}u(t) + e^{-A(t)}u'(t) - (e^{-A(t)} \cdot b(t)) = e^{-A(t)} \cdot (u'(t) - a(t)u(t) - b(t)) = 0.$$

Hence $e^{-A(t)}u(t) - B^*(t) = C \iff u(t) = e^{A(t)}(B^*(t) + C)$.

Exercise 3:

Determine a solution of the initial value problem

$$u'(t) \;=\; \cos(t)\cdot u(t) \;+\; te^{\sin(t)}, \qquad u(0)=5.$$

Solution:
With the same notation of Exercise 2) we have $a(t) = \cos(t)$ and $b(t) = te^{\sin(t)}$.

Thus we can choose $A(t) = \sin(t)$ and obtain

$e^{-A(t)}\cdot b(t) = e^{-\sin(t)}\cdot te^{\sin(t)} \;=\; t = (B^*(t))'$.

The choice $B^*(t) = \frac{t^2}{2}$ delivers

$u(t) = e^{A(t)}(B^*(t)+C) = e^{\sin(t)}(\frac{t^2}{2}+C)$.

Using the initial condition $u(0) = 5$ we find

$u(0) = e^{A(0)}(B^*(0)+C) = e^{\sin(0)}(\frac{0^2}{2}+C) = 1\cdot(0+C) = 5 \Longrightarrow C = 5$.

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