Fachbereich Mathematik der Universität Hamburg **WiSe 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 2, Exercise class

Exercise 1:

Decide which of the following differential equations are linear and which are of second order. Which of the linear equations are homogeneous and which are inhomogeneous?

a) $(y(t))^2 - (y'(t))^2 = 0\,.$

b) $y'(t) - t^2 y(t) = 0\,.$

c) $y'(t) - t^2 y(t) - e^{-t} = 0\,.$

d) $y''(t) - 2y'(t) + y(t) = t^2\,.$

e) $y''(t) + 2y'(t) - y(t)^4 = 0\,.$

Hint: you don't need to solve the differential equations!

Solution 1:

The differential equations in b), c) and d) are linear.

The differential equation in b) is homogeneous. The ones in c) and d) are inhomogeneous.

The differential equations in d) and e) are of second order.

Exercise 2:

For the differential equations b) and c) investigate whether any linear combination

$y(t) := c_1\hat{y}(t) + c_2\tilde{y}(t),\ c_1,\ c_2 \in \mathbb{R}$

of two solutions

$\hat{y} : \mathbb{R} \to \mathbb{R},\ t \mapsto \hat{y}(t)$ and $\tilde{y} : \mathbb{R} \to \mathbb{R},\ t \mapsto \tilde{y}(t)$

solve the differential equation as well.

Justify your results.

Solution:

First, the linearity of the derivative operator $\frac{\mathrm{d}}{\mathrm{dt}}$ yields

$$y'(t) = (c_1\hat{y}(t) + c_2\tilde{y}(t))' = c_1\hat{y}'(t) + c_2\tilde{y}'(t)\,.$$

If $\tilde{y}$ and $\hat{y}$ are solutions of the differential equation b), then the following equations hold

$$\hat{y}'(t) = t^2\hat{y}(t) \qquad \text{and} \qquad \tilde{y}'(t) = t^2\tilde{y}(t)\,.$$

Thus

$$y'(t) = c_1\hat{y}'(t) + c_2\tilde{y}'(t) = c_1\left(t^2\hat{y}(t)\right) + c_2\left(t^2\tilde{y}(t)\right) = t^2\left(c_1\hat{y}(t) + c_2\tilde{y}(t)\right) = t^2 y(t)\,.$$

Therefore any linear combination of solutions of the differential equation b) solves the differential equation b).

If $\tilde{y}$ and $\hat{y}$ are solutions of the differential equation c), then the following equations hold

$$\hat{y}'(t) = t^2\hat{y}(t) + e^{-t} \qquad \text{and} \qquad \tilde{y}'(t) = t^2\tilde{y}(t) + e^{-t}\,.$$

Thus

$$y'(t) = c_1\left(t^2\tilde{y}(t) + e^{-t}\right) + c_2\left(t^2\hat{y}(t) + e^{-t}\right) = t^2\left(c_1\tilde{y}(t) + c_2\hat{y}(t)\right) + (c_1 + c_2)e^{-t} = t^2 y(t) + (c_1 + c_2)e^{-t}\,.$$

Hence y is a solution of the differential equation only if $c_1 + c_2 = 1$.

Explanation: Arbitrary linear combinations of homogeneous linear differential equations are still solutions of the differential equation. In the inhomogeneous case this is valid in general only for special linear combinations.

Exercise 3:

For the differential equation a) investigate whether every linear combination $y(t) := c_1\hat{y}(t) + c_2\tilde{y}(t)$ of two solutions

$\hat{y} : \mathbb{R} \to \mathbb{R},\, t \mapsto \hat{y}(t)$ und $\tilde{y} : \mathbb{R} \to \mathbb{R},\, t \mapsto \tilde{y}(t)$

solves the differential equation. Justify your results.

Solution:

If $\tilde{y}$ and $\hat{y}$ are solutions of the differential equation a), then the following equations hold

$$(\hat{y}'(t))^2 = (\hat{y}(t))^2 \qquad \text{and} \qquad (\tilde{y}'(t))^2 = (\tilde{y}(t))^2 \,.$$

Substituting y into the differential equation yields

$$\begin{aligned}(y(t))^2 - (y'(t))^2 &= \left(c_1\hat{y}(t) + c_2\tilde{y}(t)\right)^2 - \left(c_1\hat{y}'(t) + c_2\tilde{y}'(t)\right)^2 \\ &= c_1^2\hat{y}(t)^2 + 2c_1c_2\hat{y}(t)\tilde{y}(t) + c_2^2\tilde{y}(t)^2 - \left(c_1^2\hat{y}'(t)^2 + 2c_1c_2\hat{y}'(t)\tilde{y}'(t) + c_2^2\tilde{y}'(t)^2\right) \\ &= c_1^2\underbrace{\left(\hat{y}(t)^2 - \hat{y}'(t)^2\right)}_{0} + 2c_1c_2(\hat{y}(t)\tilde{y}(t) - \hat{y}'(t)\tilde{y}'(t)) + c_2^2\underbrace{\left(\tilde{y}(t)^2 - \tilde{y}'(t)^2\right)}_{0} \\ &= 2c_1c_2 \cdot (\hat{y}(t)\tilde{y}(t) - \hat{y}'(t)\tilde{y}'(t)) \overset{!}{=} 0.\end{aligned}$$

Hence the differential equation is fulfilled if c_1 or c_2 are equal to zero. But then we don't get a proper linear combination, but only a multiple of one of the known solutions. Attention: This does not have to be the case either! Multiples of solutions of non-linear equations do not necessarily have to be solutions!

If, on the contrary $c_1c_2 \neq 0$ applies and one has a true linear combination of two different solutions, then the differential equation is not fulfilled if, for example

$\hat{y}'(t) = \hat{y}(t) \qquad$ and $\qquad \tilde{y}'(t) = -y(t)$ hold.

Since the differential equation is non-linear, one cannot expect that linear combinations of solutions are still solutions.

Not required from the students, but perhaps helpful for the discussion:

Actually $\hat{y}(t) = e^t$ and $\tilde{y}(t) = e^{-t}$ are solutions, but for $y = \hat{y} + \tilde{y}$ it holds for example:

$$(y(t))^2 - (y'(t))^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2 = (e^{2t} + 2 + e^{-2t}) - (e^{2t} - 2 + e^{-2t}) = 4 \neq 0 \,.$$

Dates of classes: 30.10-03.11.2023.