Fachbereich Mathematik der Universität Hamburg **Winter Term 2023/24**
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Differential Equations I for Students of Engineering Sciences

Sheet 1, Homework

Exercise 1: (Repetition of mathematics II, if necessary please revise!)

Determine all eigenvalues, eigenvectors and if appropriate generalized eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 1 & 0 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix} .$$

Solution:

The eigenvalues can be read directly from the diagonal.
Eigenvectors:

$$\lambda_1 = -3 : \quad \begin{pmatrix} 8 & 1 & 0 & 0 \\ 0 & 8 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{v}^{[1]} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftarrow \boldsymbol{v}^{[1]} = \begin{pmatrix} -\frac{1}{8} \\ 1 \\ 4 \\ -8 \end{pmatrix}$$

$$\lambda_2 = -1 : \quad \begin{pmatrix} 6 & 1 & 0 & 0 \\ 0 & 6 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \boldsymbol{v}^{[2]} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftarrow \boldsymbol{v}^{[2]} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 5 : \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & -8 \end{pmatrix} \boldsymbol{v}^{[3]} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longleftarrow \boldsymbol{v}^{[3]} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvectors are $k \cdot \boldsymbol{v}^{[1]}, \hat{k} \cdot \boldsymbol{v}^{[2]}, \tilde{k} \cdot \boldsymbol{v}^{[3]}$ where $k, \hat{k}, \tilde{k} \in \mathbb{R} \setminus \{0\}$. There are no other eigen directions.
It holds $a(5) - g(5) = 1$.

Generalized eigenvector of first level of $\lambda_3 = 5, \boldsymbol{v}^{[4]}$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & -8 \end{pmatrix} \boldsymbol{v}^{[4]} = \boldsymbol{v}^{[3]} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \boldsymbol{v}^{[4]} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad (+\tilde{k}\, \boldsymbol{v}^{[3]})$$

Exercise 2: (Repetition of mathematics II, if necessary please revise!)

Please use Laplace expansion along rows or columns. Do not work solely with the rule of Sarrus!

a) Let A be a real $n \times n$ matrix $\lambda = a + ib$ with $a, b \in \mathbb{R}$ an eigenvalue of A with corresponding eigenvector $\boldsymbol{v}$ and i the imaginary unit with $i^2 = -1$. Show that $\bar{\lambda} = a - ib$ is an eigenvalue of A with corresponding eigenvector $\bar{\boldsymbol{v}}$.

b) Determine all eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 2 \\ -3 & 0 & 2 \end{pmatrix}.$$

Solution:

a) $\lambda \cdot \boldsymbol{v} = A \cdot \boldsymbol{v} \Longleftrightarrow \overline{\lambda \cdot \boldsymbol{v}} = \overline{A \cdot \boldsymbol{v}} \Longleftrightarrow \bar{\lambda} \cdot \bar{\boldsymbol{v}} = \bar{A} \cdot \bar{\boldsymbol{v}} = A \cdot \bar{\boldsymbol{v}}$

b)

$$\det(B - \lambda E) = \begin{vmatrix} 2-\lambda & 0 & 3 \\ 2 & 1-\lambda & 2 \\ -3 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 3 \\ -3 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(2-\lambda)^2 + 9].$$

Hence, the eigenvalues of the matrix are

$$\lambda_1 = 1, \quad \lambda_{2,3} = 2 \pm 3i\,.$$

For the eigenvalue 1 one computes the eigenvector

$$\begin{pmatrix} 2-1 & 0 & 3 \\ 2 & 1-1 & 2 \\ -3 & 0 & 2-1 \end{pmatrix} \boldsymbol{v} = 0 \Longleftarrow \boldsymbol{v} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}.$$

For the eigenvalue $2 - 3i$ one computes the eigenvector

$$\begin{pmatrix} 3i & 0 & 3 \\ 2 & 3i-1 & 2 \\ -3 & 0 & 3i \end{pmatrix} \boldsymbol{w} = 0 \Longleftrightarrow \begin{cases} w_3 = -iw_1 \\ 2w_1 + (3i-1)w_2 - 2iw_1 = 0 \\ -3w_1 + 3i(-iw_1) = 0 \end{cases}$$

From the second line one computes $w_2 = \dfrac{2(1-i)w_1}{1-3i}$.

The choice $w_1 := 1 - 3i$ yields the eigenvector:

$$w = \begin{pmatrix} 1-3i \\ 2-2i \\ -3-i \end{pmatrix}$$

According to Part a) $\bar{\boldsymbol{w}}$ is an eigenvector of $2 + 3i$.

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