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Differential Equations I
for Students of Engineering Sciences

Sheet 1, Exercise class

Exercise 1:

Let $y(t)$ be the number of common vole in a given region at time t. In a very simple model it is assumed that the increase of the number of mice per time unit is proportional to the number of mice.

- Derive a difference equation which approximately describes the development of the number of mice in a short period of time.
- Describe the development of the number of mice using a differential equation.
- Can you determine the number of mice at time $t = 10$ (depending on some proportionality factor, of course)?

  Which information do you miss?

Sketch of solution of Exercise 1:

$$y(t + \Delta t) - y(t) \approx \lambda y(t) \cdot \Delta t \iff \frac{y(t + \Delta t) - y(t)}{\Delta t} \approx \lambda y(t) .$$

If y is differentiable one obtains

$$y'(t) = \lim_{\Delta t \to 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} = \lambda y(t) .$$

We already know a function which only changes by a constant factor when differentiated, namely $y^*(t) = e^{\lambda t}$.

Any multiple of y^* also has this property. Thus we can choose $y(t) = ke^{\lambda t}$ with a constant k.

We cannot determine the value of y at a given time without knowing k. This is clear! Without knowing the initial population, we can hardly expect to infer the number of mice at any later time from the growth rate.

If, on the other hand we know $y(0)$, for example $y(0) = 50$, then $k = 50$ and $y(10) = 50e^{10\lambda}$.

Exercise 2:

We look for functions $y : \mathbb{R} \to \mathbb{R},\, t \mapsto y(t)$ with

$$y'''(t) + 2y''(t) - y'(t) - 2y(t) = 0\,,$$

i.e. solutions of the differential equation

$$y''' + 2y'' - y' - 2y = 0\,.$$

With the help of the ansatz $y(t) = ke^{\lambda t}$, k, λ constant, determine solutions of this differential equation different from the zero function.

Solution of Exercise 2:

Inserting the ansatz into the differential equation delivers:

$$k\lambda^3 e^{\lambda t} + 2k\lambda^2 e^{\lambda t} - k\lambda^1 e^{\lambda t} - 2ke^{\lambda t} = 0$$

Thus the equation is satisfied if

$$k = 0 \implies y \equiv 0 \text{ (zero function)}$$

or

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = 0\,.$$

Since the sum of the coefficients vanishes, $\lambda = 1$ is a zero of the polynomial.

Polynomial division yields:

$$\lambda^3 + 2\lambda^2 - \lambda - 2 = (\lambda - 1)(\lambda^2 + 3\lambda + 2) = (\lambda - 1)(\lambda + 1)(\lambda + 2).$$

Therefore for $\lambda \in \{-2, -1, 1\}$ one finds the solutions

$y_1(t) = k_1 e^{-2t}$, $y_2(t) = k_2 e^{-t}$ and $y_3(t) = k_3 e^t$

of the differential equation.

The constants k_1, k_2, k_3 can then be chosen arbitrarily. Of course, choosing zero one gets the trivial solution $y \equiv 0$.

Hint for tutors: The fast students can perhaps already be motivated here to convince themselves that

- for example $y_4 = y_1 + y_2$ and $y_5 = y_1 - y_3$ are solutions as well.
- any linear combination of solutions of the differential equation is also a solution.

It is essential to point out that a systematic discussion of these issues will occur later during the lectures, and that not for every differential equation linear combinations of solutions are solutions. In fact, this property characterizes linear differential equations.

Dates of classes: 16.10.-20.10.2023