**Fachbereich Mathematik der Universität Hamburg** **WiSe 2023/24**
Prof. Dr. T. Schmidt, Dr. H. P. Kiani, E. Ficola

# Differential Equations I for Students of Engineering Sciences

## Sheet 7, Exercise class

To shorten the notation we employ the Doetsch symbol:

$$F(s) := \mathscr{L}f(s) := \int_0^\infty e^{-st} f(t)dt \iff f(t) \circ\!\!-\!\!\bullet\, F(s)$$

The following correspondences or relations for $\mathrm{Re}\,(s) > \gamma$ , which were either proved in the lecture or could be proved completely analogously to the lectures procedure, may be used. We always have $f(t) = 0, \forall t < 0$ .

| $f(t)$, $t \geq 0$ | $F$ | $\gamma$ |
|---|---|---|
| 1 i.e. $h_0(t)$ | $\frac{1}{s}$ | 0 |
| $h_a(t)$ | $e^{-as}\frac{1}{s}$ | 0 |
| $t^n, \quad n \in \mathbb{N}$ | $\frac{n!}{s^{n+1}}$ | 0 |
| $e^{at}, \quad a \in \mathbb{C}$ | $\frac{1}{s-a}$ | $\mathrm{Re}\,(a)$ |
| $e^{at}\sin(\omega t), \quad \omega \in \mathbb{R}$ | $\frac{\omega}{(s-a)^2+\omega^2}$ | 0 |
| $e^{at}\cos(\omega t), \quad \omega \in \mathbb{R}$ | $\frac{s-a}{(s-a)^2+\omega^2}$ | 0 |

$h_a(t)$ for $a \geq 0$ is defined as follows: $h_a(t) := \begin{cases} 1 & t \geq a \geq 0, \\ 0 & t < a. \end{cases}$

If $f(t) \circ\!\!-\!\!\bullet\, F(s)$ , then the following shifting theorems hold.

$I)$ $h_a(t)f(t-a) \circ\!\!-\!\!\bullet\, e^{-sa}F(s)$ — Shifting in the original space
Multiplication with exponential function in the image space

$II)$ $e^{at}f(t)$, $a \in \mathbb{C}$ $\circ\!\!-\!\!\bullet\, F(s-a)$ — Shifting in
the image space/ Multiplication with
exponential function in the original space

**Exercise 1:**

a) Which is the algebraic equation resulting from Laplace transformation of the initial value problem

$$u'' - 2u' + u = \sin(4t) + 2te^{-t}, \text{ for } t > 0, \qquad u(0) = 1, u'(0) = 0?$$

Please justify your answer by intermediate computations.
Compute the solution of the algebraic equation.

b) Let $F(s) = \dfrac{1}{s(s+1)^2}$ be the Laplace transform of the function

$$f : \mathbb{R}_0^+ \to \mathbb{R}, \quad f : t \mapsto f(t).$$

Determine $f(t)$.

**Exercise 2:**

a) For the following matrices $\boldsymbol{A}$ analyse the stability of the stationary point $(0,0)^T$ of the linear system $\boldsymbol{u}'(t) = A\,\boldsymbol{u}(t)$.

**i)** $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, **ii)** $A = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$, **iii)** $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

b) Consider the following linear system

$$\dot{\boldsymbol{u}}(t) = \begin{pmatrix} -3 & 0 & 3 \\ -1 & -\gamma & 1 \\ 3 & 0 & -3 \end{pmatrix} \boldsymbol{u}(t).$$

Determine the stability behaviour of the stationary point $(0,0,0)^T$ depending on the parameter $\gamma \in \mathbb{R}$.

**Dates of classes:** 22.01.-26.01.2024