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Differential Equations I for Students of Engineering Sciences
Sheet 5, Homework

Exercise 1)

Consider the following fourth-order differential equation

$$u^{(4)}(t) + a_3 u^{'''}(t) + a_2 u^{''}(t) + a_1 u'(t) + a_0 u(t) = 0 \tag{1}$$

with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For each of the following sets of functions, determine whether they can be (with suitable coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$) a fundamental system for the solution space of the differential equation.

Justify your answers.

a) $M_1 := \{u_1(t) = e^t,\ u_2(t) = e^{5t},\ u_3(t) = e^{9t}\}$.

b) $M_2 := \{u_1(t) = e^t,\ u_2(t) = e^{it},\ u_3(t) = e^{2t},\ u_4(t) = e^{2it}\}$.

c) $M_3 := \{u_1(t) = 1,\ u_2(t) = t,\ u_3(t) = e^{2t},\ u_4(t) = e^{-2t}\}$.

d) $M_4 := \{u_1(t) = e^t,\ u_2(t) = \sin(2t),\ u_3(t) = e^{-2it},\ u_4(t) = e^{2it}\}$.

Exercise 2)

Consider the differential equation

$$u''(t) + 9u(t) = b(t)$$

a) Determine a real representation for the general solution of the corresponding homogeneous differential equation.

b) Compute the solutions of the differential equation for the inhomogeneities
i) $b(t) = 5e^{-t}$, **ii)** $b(t) = 5\sin(2t)$, **iii)** $b(t) = 5\sin(3t)$.

c) Determine the solution of the corresponding initial value problems for the initial values

$$u(0) = u'(0) = 0\,.$$

In each case check whether the solutions are bounded for $t \geq 0$ and whenever possible provide upper bounds for $|u(t)|,\ t \geq 0$.

Exercise 3) Somewhat more demanding.

We look for a particular solution u_p of the inhomogeneous differential equation

$$\mathcal{L}_0[u] := \sum_{k=0}^{m} a_k u^{(k)}(t) = b(t) = b_0 e^{\alpha t}, \qquad a_m = 1, a_k \in \mathbb{R},\, 0 \neq b_0 \in \mathbb{R},\, \alpha \in \mathbb{C}.$$

a) Prove that the ansatz $u_p(t) = Be^{\alpha t}$, $B \in \mathbb{C}$ is successful if and only if α is not a root of the characteristic polynomial

$$P_0(\lambda) := \sum_{k=0}^{m} a_k \lambda^k.$$

b) Prove that the ansatz $u_p(t) = Bte^{\alpha t}$, $B \in \mathbb{C}$ is successful if α is a simple root of the characteristic polynomial

$$P_0(\lambda) := \sum_{k=0}^{m} a_k \lambda^k.$$

Hint: Use the factorization from page 40 of the lecture.

c) Let now α be a root of the characteristic polynomial with multiplicity $l \in \mathbb{N}$, $l \geq 2$. Thus

$$P_0(\lambda) = P_l(\lambda)(\lambda - \alpha)^l,\ P_l(\alpha) \neq 0$$

and

$$\mathcal{L}_0[u] := \mathcal{L}_l\left[\left(\frac{d}{dt} - \alpha\right)^l u\right].$$

Prove that $u_p(t) := Bt^l e^{\alpha t}$, $B \in \mathbb{C}$ is an appropriate ansatz for a fundamental solution of the differential equation.

Hints:

Define P_j and $\mathcal{L}_j$ by

$P_0(\lambda) = P_j(\lambda)(\lambda - \alpha)^j$, $\mathcal{L}_0[u] = \mathcal{L}_j\left[\left(\frac{d}{dt} - \alpha\right)^j u\right]$, $\qquad j = 0, 1, 2, \ldots, l.$

α is a root of multiplicity $(l - j)-$ of P_j . In particular it is not a zero of P_l .

Show

$$\mathcal{L}_0[Bt^l e^{\alpha t}] = \frac{l!}{(l-j)!} B\mathcal{L}_j[t^{l-j} e^{\alpha t}]$$

by induction and using the factorization method on page 40 of the lecture.

Hand in until: 15.12.2023