Mathematics Department
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WiSe 2023/24

Mathematics III Exam
(Module: Differential Equations I)

March 4, 2024

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Assessment according to examin. reg: with Analysis III ☐ single scoring ☐

I was instructed about the fact that the exam performance will only be assessed if the Central Examination Office of TUHH verifies my official admission before the exam's beginning in retrospect.

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Exercise	**Points**	**Evaluator**
1		
2		
3		
4		

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Exercise 1: (5 points)

Compute the general solution of the following differential equation

$$u'(t) = \sin(2t)u(t) + e^{\cos(2t)}(u(t))^3.$$

Hint: It is useful to employ a standard substitution.

Solution :

The differential equation is a Bernoulli equation.

For $\alpha = 3$, $a(t) = \sin(2t)$ and $b(t) = e^{\cos(2t)}$ and $y = u^{1-\alpha} = u^{-2}$, we obtain for y the linear differential equation

$$y'(t) = (1-\alpha)(a(t)y(t) + b(t)) = -2(\sin(2t)y(t) + e^{\cos(2t)}).$$ **(1 point)**

Alternative 1) Variation of constants

By separation, we compute for the corresponding inhomogeneous differential equation:

$y_h' = -2\sin(2t)y_h \implies y_h(t) = e^{\int -2\sin(2t)dt} = Ce^{\cos(2t)}$. **(1,5 points)**

Variation of constants:

$y_p(t) = C(t)e^{\cos(2t)} \overset{\text{DE}}{\longrightarrow} C'(t)e^{\cos(2t)} \overset{!}{=} -2e^{\cos(2t)} \longrightarrow c'(t) = -2.$

The choice $C(t) = -2t$, for example, yields

$y(t) = y_h(t) + y_p(t) = (C - 2t)e^{\cos(2t)}$. **(1,5 points)**

Alternative 2) Formula in lecture 1

$y(t) = e^{A(t)}(B^*(t) + C)$

where $A'(t) = \hat{a}(t)$, $(B^*(t))' = e^{-A(t)} \cdot \hat{b}(t)$, $C \in \mathbb{R}$. **Ansatz: (1 point)**

$A'(t) = \hat{a}(t) = -2(\sin(2t)$ for example $A(t) = \cos(2t)$.

With $\hat{b}(t) = -2e^{\cos(2t)}$, we calculate

$e^{-A(t)} \cdot \hat{b}(t) = e^{-cos(2t)} \cdot (-2e^{\cos(2t)}) = -2 = (B^*(t))'$.

For $B^*(t) = -2t$, for example, we obtain

$y(t) = y(t) = e^{A(t)}(B^*(t) + C) = e^{\cos(2t)}(-2t + C)$. **(2 points)**

Back substitution: Due to $y = u^{1-\alpha} = u^{-2}$, we get

$u(t) = \pm\frac{1}{\sqrt{y}} = \pm e^{-\frac{\cos(2t)}{2}}(C - 2t)^{-\frac{1}{2}}$. **(1 point)**

Exercise 2: (4 points)

Consider the initial value problem

$$u'''(t) - 5u''(t) + 2u(t) = 3 + \cos(t)\,, \qquad u(0) = 4,\ u'(0) = 3,\ u''(0) = 0.$$

a) What is the order of the differential equation?

b) Is it an explicit equation? If this is not the case, provide an equivalent explicit differential equation.

c) Reformulate the initial value problem as an equivalent initial value problem for a system of first order.

Solution:

a) The differential equation is of order three. **(1 point)**

b) No. Explicit form:

$$u'''(t) = 5u''(t) - 2u(t)) + 3 + \cos(t). \qquad (\textbf{1 point})$$

c) For

$$\boldsymbol{y} := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}, \qquad \boldsymbol{y}' = \begin{pmatrix} u' \\ u'' \\ u''' \end{pmatrix}$$

we obtain the equivalent system **(2 points)**

$$\boldsymbol{y}' = \begin{pmatrix} u' \\ u'' \\ u''' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 5 \end{pmatrix} \begin{pmatrix} u \\ u' \\ u'' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 + \cos(t) \end{pmatrix}, \qquad \boldsymbol{y}(0) := \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}.$$

respectively

$$\boldsymbol{y}' = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 3 + \cos(t)) \end{pmatrix}, \qquad \boldsymbol{y}(0) := \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}.$$

Exercise 3: (4 points)

Consider the following differential equation of order three

$$u'''(t) + a_2u''(t) + a_1u'(t) + a_0u(t) = 0 \qquad (*)$$

with real coefficients $a_0, a_1, a_2 \in \mathbb{R}$. Examine for each of the following sets of functions, if it might be (for suitable coefficients $a_0, a_1, a_2 \in \mathbb{R}$) a fundamental system for the solution space of the differential equation.

Justify your answers.

a) $M_1 := \{u_1(t) = -t,\, u_2(t) = 1,\, u_3(t) = 2t\}$.

b) $M_2 := \{u_1(t) = e^{-t},\, u_2(t) = e^{t},\, u_3(t) = e^{2t},\, u_4(t) = e^{3t}\}$.

c) $M_3 := \{u_1(t) = e^{-t},\, u_2(t) = e^{it},\, u_3(t) = e^{2it}\}$.

d) $M_4 := \{u_1(t) = 1,\, u_2(t) = e^{-2it},\, u_3(t) = e^{2it}\}$.

Solution: (1+1+1+1 points)

The solution space is of dimension three.

a) M_1 cannot be a fundamental system as it holds, for example, that $u_3(t) + 2u_1(t) = 0$. The space that is spanned by the functions in M_1 is only of dimension two.

b) As the solution space if of dimension three, the set M_2 cannot be a fundamental system for (1). The space that is spanned by the functions in M_2 is of dimension four.

c) Complex-valued solutions of linear differential equations with real-valued coefficients always appear in conjugate complex pairs! Therefore, $M3$ cannot be a fundamental system for (1).

d) M_4 is a fundamental system for $(*)$ with suitable coefficients.

Not required from the students: The characteristic polynomial is $P(\lambda) = \lambda(\lambda^2 + 4)$, and the corresponding differential equation $u'''(t) + 4u'(t) = 0$.

Exercise 4 (7 points)

Consider the system of differential equations

$$\boldsymbol{u}'(t) = \begin{pmatrix} 3 & -5 \\ 5 & -5 \end{pmatrix} \boldsymbol{u}(t)\,.$$

a) Analyse the stability of the stationary point $(0,0)^T$ of the system.

b) Determine a real-valued fundamental system of the system of differential equations.

Solution:

a) Computation of eigenvalues

$$\det \begin{pmatrix} 3-\lambda & -5 \\ 5 & -5-\lambda \end{pmatrix} = (3-\lambda)(-5-\lambda) + 25 = \lambda^2 + 2\lambda - 15 + 25.$$

The eigenvalues of the system matrix are given by

$$(\lambda+1)^2 + 9 = 0 \iff (\lambda+1)^2 = -9 \iff \lambda + 1 = \pm 3i \implies \lambda_{1,2} = -1 \pm 3i\,.$$

(2 points)

The real part of all eigenvalues is negative. The stationary point is asymptotically stable. **(1 points)**

b) The eigenvector corresponding to $\lambda_1 = -1+3i$ can be obtained by solving the system of equations

$$\begin{pmatrix} 4-3i & -5 \\ 5 & -4-3i \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

First row: $(4-3i)z_1 - 5z_2 = 0 \iff z_2 = \frac{(4-3i)z_1}{5}$

We choose, for example, $\boldsymbol{z} = \binom{5}{4-3i}$.

Then, the second row is fulfilled as well.

A complex-valued solution is: $\boldsymbol{z}^{[1]}(t) = e^{(-1+3i)t}\,\boldsymbol{z} = e^{-1t}e^{3it}\,\boldsymbol{z}\,.$ **(2 points)**

A real-valued fundamental system, for example, is

$FM(t) = \left(\boldsymbol{u}^{[1]}(t),\, \boldsymbol{u}^{[2]}(t)\right) = (\operatorname{Re}(\boldsymbol{z}^{[1]}(t)), \operatorname{Im}(\boldsymbol{z}^{[1]}(t)))\,.$

Due to

$$\boldsymbol{z}^{[1]}(t) = e^{-t}(\cos(3t)+i\cdot\sin(3t))\cdot\begin{pmatrix} 5 \\ 4-3i \end{pmatrix} = e^{-t}\begin{pmatrix} 5\cos(3t) + 5i\cdot\sin(3t) \\ 4\cos(3t) + 4i\cdot\sin(3t) - 3i\cos(3t) + 3\sin(3t) \end{pmatrix}$$

we obtain

$$\boldsymbol{u}^{[1]}(t) = e^{-t}\begin{pmatrix} 5\cos(3t) \\ 4\cos(3t) + 3\sin(3t) \end{pmatrix}, \quad \boldsymbol{u}^{[2]}(t) = e^{-t}\begin{pmatrix} 5\sin(3t) \\ 4\sin(3t) - 3\cos(3t) \end{pmatrix}.$$

(2 points)