Fachbereich Mathematik der Universität Hamburg **Winter Term 2022/23**
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Differential Equations I for Students of Engineering Sciences

Sheet 6, Exercise class

In order to shorten the notation we use the Doetsch symbol:

$$F(s) := L\,[f(t)] := \int_0^\infty e^{-st} f(t)dt \iff f \circ\!\!-\!\!\bullet\, F$$

One may employ the following correspondences for $\operatorname{Re}(s) > \gamma$, which have either been proven in the lecture or can be proved completely analogously to the procedure of the lecture. Let always $f(t) = 0, \forall t < 0$.

$f(t),\, t \geq 0$	F	γ
1 d.h. $h_0(t)$	$\frac{1}{s}$	0
$h_a(t)$	$e^{-as}\frac{1}{s}$	0
$t^n, \quad n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	0
$e^{at}, \quad a \in \mathbb{C}$	$\frac{1}{s-a}$	$\operatorname{Re}(a)$
$\sin(\omega t), \quad \omega \in \mathbb{R}$	$\frac{\omega}{s^2+\omega^2}$	0
$\cos(\omega t), \quad \omega \in \mathbb{R}$	$\frac{s}{s^2+\omega^2}$	0

Where $\quad h_a(t) := \begin{cases} 1 & t \geq a \geq 0, \\ 0 & t < a. \end{cases}$

If $f \circ\!\!-\!\!\bullet\, F$, then the following shifting properties hold.

$I)\quad h_a(t)f(t-a) \quad \circ\!\!-\!\!\bullet \quad e^{-sa}F(s)$ — Shifting in the original space / Mult. with exp function in the image space

$II)\quad e^{at}f(t),\ a \in \mathbb{C} \quad \circ\!\!-\!\!\bullet \quad F(s-a)$ — Shifting in the image space/ Mult. with exp function in the original space

Exercise 1:

a) Consider the initial value problem

$$y'' - 2y' + y = \sin(4t) + 2te^{-t}, \text{ for } t > 0, \qquad y(0) = 1, y'(0) = 0.$$

Into which algebraic equation can the initial value problem be transformed by the Laplace transformation?

Please justify your answer with intermediate calculations.

Compute the solution of the algebraic equation.

b) Let $F(s) = \dfrac{1}{s(s+1)^2}$ be the Laplace transform of the function

$$f : \mathbb{R}^+ \to \mathbb{R}, \quad f : t \mapsto f(t).$$

Determine $f(t)$.

Exercise 2:

Solve the following initial value problems using the Laplace transform.

a) $y'' + 9y = h_1(t) - h_2(t) \qquad y(0) = y'(0) = 0$

b) (Only for very fast students) $\qquad y'' + y = e^{-t}\sin(2t)$, $y(0) = \alpha$, $y'(0) = \beta$.

Dates of classes: 09.01.-13.01.2023