Fachbereich Mathematik der Universität Hamburg **Winter Term 2022/23**
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Differential Equations I for Students of Engineering Sciences

Sheet 6, Homework

Exercise 1:

Consider the initial value problem

$$x^2y''(x) - 2x^2y'(x) + (x^2-2)y(x) = 0\,, \qquad y(0) = y'(0) = 0\,.$$

Prove that with y every multiple of y is also a solution of the initial value problem.

To compute a solution use a power series ansatz

$$y(x) = \sum_{k=0}^{\infty} a_k x^k\,.$$

a) Determine a recursion formula to compute the coefficients a_k.

b) Prove that with the additional condition $a_2 = 1$ one finds

$$a_k = \frac{1}{(k-2)!} \qquad \forall k \geq 2.$$

Which function does the series represent?

Exercise 2:

a) Consider the initial value problem

$$y'(t) + y(t) = b(t), \qquad b(t) := \begin{cases} 0 & t < 0 \\ \sin(t) & 0 \leq t \leq \frac{\pi}{2} \\ 1 & t > \frac{\pi}{2} \end{cases}, \qquad y(0) = 0. \tag{1}$$

(i) Draw $(t, b(t))$ for $t \in [-2, 4]$ and explain why the following equality holds for all $t \geq 0$.

$$b(t) = \sin(t) - h_{\frac{\pi}{2}}(t)\cos(t - \frac{\pi}{2}) + h_{\frac{\pi}{2}}(t)\,. \tag{2}$$

(ii) Determine the Laplace transform of b by

1. applying directly the definition

$$B(s) := L\,[b(t)] := \int_0^\infty e^{-st} b(t)dt\,.$$

   Hint: $\displaystyle\int e^{\alpha t} \cdot \sin(t)\,dt \;=\; \frac{e^{\alpha t}}{\alpha^2+1}\,(\alpha \cdot \sin(t) - \cos(t)) + C\,.$

2. using (2) and the table or calculation rules for the Laplace transformation.

(iii) Solve the initial value problem (1) using Laplace transformation.

b) Solve the following initial value problems using Laplace transformation.

$$\begin{aligned} u'' - 2(v-u) &= 1 \qquad & u(0) = v(0) = 0 \\ v'' + 2(v-u) &= 0 \qquad & u'(0) = v'(0) = 1. \end{aligned}$$

Hand in until: 13.01.2023