Mathematics Department **WiSe 2022/2023**
Prof. Dr. J. Behrens

Mathematics III Exam
(Module: Differential Equations I)

March 6, 2023

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Wertung nach PO: together with Analysis III ☐ single scoring ☐

I was instructed about the fact that the exam performance will only be assessed if the Central Examination Office of TUHH verifies my official admission before the exam's beginning in retrospect.

(Signature)

Exercise	**Points**	**Evaluator**
1		
2		
3		
4		

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Exercise 1 (3 + 4 points)

Determine the solutions of the following initial value problems

a)

$$y'(x) = \frac{1+\cos(x)}{(y(x))^2} \qquad \text{for } x > 0, \qquad y(0) = 3.$$

b)

$$x^2 y''(x) - xy'(x) - 8y(x) = 0 \qquad \text{for } x > 1, \qquad y(1) = 0,\ y'(1) = 6.$$

Solution:

a) It is a separable variable differential equation. For $y \neq 0$ one computes
$\frac{dy}{dx} = \frac{1+\cos(x)}{y^2} \iff y^2 dy = (1+\cos(x))dx.$ **(1 point)**
And from this
$\int y^2 dy = \int (1+\cos(x))dx \iff \frac{y^3}{3} = x + \sin(x) + C$

$\iff y(x) = \sqrt[3]{3x + 3\sin(x) + 3C}.$ **(1 point)**

$y(0) = \sqrt[3]{0 + 3\sin(0) + 3C} \overset{!}{=} 3 \iff C = 9\,.$ **(1 point)**

Thus $y(x) = \sqrt[3]{3x + 3\sin(x) + 27}\,.$

b) It is an Euler differential equation. With the ansatz $y(x) = x^r$ one obtains

$$x^2 \cdot r(r-1)x^{r-2} - x \cdot rx^{r-1} - 8x^r = x^r \cdot (r^2 - 2r - 8) \overset{!}{=} 0\,.$$

Therefore for $x > 0$:

$$r^2 - 2r - 8 \overset{!}{=} 0 \iff (r-1)^2 - 9 \overset{!}{=} 0 \iff r - 1 = \pm\sqrt{9} \iff r = -2 \lor r = 4.$$

The general solution is: $y(x) = c_1 x^{-2} + c_2 x^4\,.$ **(2 points)**

From $y(1) = c_1 + c_2 \overset{!}{=} 0$ we get $c_1 = -c_2\,.$
With $y'(x) = -2c_1 x^{-3} + 4c_2 x^3$ then the second initial value returns $y'(1) = -2c_1 + 4c_2 = 6c_2 \overset{!}{=} 6 \iff c_2 = -c_1 = 1$ **(2 points)**
and
$y(x) = x^4 - x^{-2}\,.$

Exercise 2 (4 points)

Consider the system of differential equations

$$\boldsymbol{y}'(t) = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} \boldsymbol{y}(t) \,.$$

Determine a real fundamental system and a real representation of the general solution of the system of differential equations.

Solution 2:

$$\det \begin{pmatrix} -\lambda & -3 \\ 3 & -\lambda \end{pmatrix} = \lambda^2 + 9.$$

The eigenvalues of the system matrix are given by

$$\lambda^2 = -9 \iff \lambda = \pm 3i \,. \qquad \textbf{(1 point)}$$

The eigenvector corresponding to $\lambda_1 = 3i$ is obtained as solution of the system of equations

$$\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Any vector with $z_2 = -iz_1$ solves the system. Thus, for example we may choose $(1, -i)^T$. The complex conjugate vector is an eigenvector for $\lambda_2 = -3i$.

With this we obtain the complex fundamental system

$$\boldsymbol{u}(t) = e^{3it} \begin{pmatrix} 1 \\ -i \end{pmatrix} , \qquad \boldsymbol{v}(t) = e^{-3it} \begin{pmatrix} 1 \\ +i \end{pmatrix} .$$

A real fundamental system is for example given by

$\boldsymbol{Y}(t) := \left(\boldsymbol{y}^{[1]}(t), \boldsymbol{y}^{[2]}(t)\right) = (\operatorname{Re}(\boldsymbol{u}(t)), \operatorname{Im}(\boldsymbol{u}(t)))$. **(1 point)**

Since $\boldsymbol{u}(t) = (\cos(3t) + i \cdot \sin(3t)) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos(3t) + i \cdot \sin(3t) \\ -i\cos(3t) + \sin(3t) \end{pmatrix}$

one gets

$$\boldsymbol{y}^{[1]}(t) = \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} , \qquad \boldsymbol{y}^{[2]}(t) = \begin{pmatrix} \sin(3t) \\ -\cos(3t) \end{pmatrix}$$

and the general solution

$\boldsymbol{y}(t) = c_1 \boldsymbol{y}^{[1]}(t) + c_2 \boldsymbol{y}^{[2]}(t)$, $\qquad c_1, c_2 \in \mathbb{R}$ or $\mathbb{C}$. **(2 points)**

Exercise 3: (5 points)

Consider the following fourth-order differential equation

$$y^{(4)}(t) + a_3 y'''(t) + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0 \tag{1}$$

with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For each of the following sets of functions, determine whether they can be (with suitable coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$) a fundamental system for the solution space of the differential equation.

Justify your answers.

a) $M_1 := \{y_1(t) = e^t,\ y_2(t) = e^{5t},\ y_3(t) = e^{9t}\}$.

b) $M_2 := \{y_1(t) = e^t,\ y_2(t) = e^{it},\ y_3(t) = e^{2t},\ y_4(t) = e^{2it}\}$.

c) $M_3 := \{y_1(t) = 1,\ y_2(t) = t,\ y_3(t) = e^{2t},\ y_4(t) = e^{-2t}\}$.

d) $M_4 := \{y_1(t) = e^t,\ y_2(t) = \sin(2t),\ y_3(t) = e^{-2it},\ y_4(t) = e^{2it}\}$.

Solution: (1+1+1+2 points)

a) Since the space of solutions has dimension four, M_1 cannot be a fundamental system for (1).

b) Complex solutions of linear differential equations with constant real coefficients always occur in conjugated complex pairs! Therefore M_2 cannot be a fundamental system for (1).

c) M_3 is a fundamental system for (1) with appropriate coefficients.

   Not required from the students: the characteristic polynomial would be $P(\lambda) = \lambda^2(\lambda^2 - 4)$, thus the differential equation $y''''(t) - 4y''(t) = 0$.

d) M_4 cannot be a fundamental system, since for example it holds:

   $e^{2it} - e^{-2it} = 2i\sin(2t)$. The space spanned by the functions in M_4 has only dimension three.

Exercise 4: (4 points)

Without knowing the value of $\gamma \in \mathbb{R}$, determine for any of the following matrices $\boldsymbol{A}_k$, $k = 1, 2, 3$, whether the zero solution is a stable or unstable stationary point (equilibrium point) of the system of differential equations $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_k \boldsymbol{x}(t)$.

Justify your answers.

i) $\boldsymbol{A}_1 = \begin{pmatrix} -2 & \gamma & 0 \\ 0 & 0 & \gamma \\ 0 & 0 & -2 \end{pmatrix}$, **ii)** $\boldsymbol{A}_2 = \begin{pmatrix} \gamma & -1 & 0 \\ 1 & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$, **iii)** $\boldsymbol{A}_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\gamma \\ 0 & \gamma & 0 \end{pmatrix}$.

Solution: (1+1+2 points)

The matrix $\boldsymbol{A}_1$ has the simple eigenvalue zero and the double eigenvalue -2. The equilibrium points are stable.

The matrix $\boldsymbol{A}_2$ has among others the eigenvalue 1. All points of equilibrium are unstable.

The matrix $\boldsymbol{A}_3$ has the characteristic polynomial

$P(\lambda) = (-1 - \lambda)(\lambda^2 + \gamma^2)$

and thus the eigenvalues $-1, -i\gamma, i\gamma$.

For $\gamma \neq 0$ there is no eigenvalue with positive real part. The eigenvalues with zero real part are simple. All points of equilibrium are stable.

For $\gamma = 0$ there is no eigenvalue with positive real part. The eigenvalue $\lambda = 0$ has algebraic multiplicity two. To check the geometric multiplicity one computes

$$(A - \lambda I)\,\boldsymbol{v} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{v} = \boldsymbol{0}$$

and obtains for example the eigenvectors $(0,1,0)^T$ and $(0,0,1)^T$. The geometric and algebraic multiplicity of the eigenvalue with real part zero coincide. Therefore the stationary points are stable.