Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III for Engineering Students
Work sheet 7

Exercise 1:

a) Let $\boldsymbol{f}$ be the vector field $\boldsymbol{f}(x,y) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$, $\mathbf{c}_1$ be the curve with the parametrization

$$\mathbf{c}_1(t) = (t, \sin(t)) \qquad t \in [0, \pi]$$

and c_2 be the mathematically positively oriented edge of the rectangle

$$R = \{(x,y) \ : \ x \in [0,1], \ y \in [0,2]\} \ = \ [0,1] \times [0,2] \cdot$$

(i) Does vector field f have a potential?

(ii) For $i = 1, 2$ compute the line integrals

$$\int_{\mathbf{c}_i} \boldsymbol{f}(x,y)\, d(x,y) \cdot$$

(iii) Compute the flow of $\boldsymbol{f}$ through R.

b) Let $\tilde{\boldsymbol{f}}$ be the vector field $\tilde{\boldsymbol{f}}(x,y) = \begin{pmatrix} x^2 - y^3 \\ y^2 + x^3 \end{pmatrix}$, $\mathbf{c}_2$ be defined as above and

$$\mathbf{c}_3(t) = (1, t^2) \qquad t \in [0, 3].$$

Compute the line integrals

$$\int_{\mathbf{c}_2} \tilde{\boldsymbol{f}}(x,y)\, d(x,y)\,, \qquad \int_{\mathbf{c}_3} \tilde{\boldsymbol{f}}(x,y)\, d(x,y) \cdot$$

Solution

a) (i) $\text{rot } \boldsymbol{f} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0.$

We compute a potential:

$$f_1(x,y) = x^2 = \phi_x(x,y) \implies \phi(x,y) = \frac{1}{3}x^3 + C(y)$$

$$f_2(x,y) = y^2 = 0 + C_y(y) = \phi_y(x,y) \implies C_y(y) = y^2, \text{ e. g. } C(y) = \frac{1}{3}y^3$$

$$\implies \phi(x,y) = \frac{1}{3}(x^3 + y^3).$$

(ii) $\int\limits_{c_1} \boldsymbol{f}\; d(x,y) = \phi\left(c_1(\pi)\right) - \phi\left(c_1(0)\right) = \phi\begin{pmatrix}\pi\\0\end{pmatrix} - \phi\begin{pmatrix}0\\0\end{pmatrix} = \frac{\pi^3}{3}.$

Since c_2 is closed, it holds $\oint\limits_{c_2} f\, d(x,y) = 0.$

(iii) For the flow through R it holds

$$F = \int_0^1 \int_0^2 \text{div } \boldsymbol{f}(x,y)\, dy\, dx = \int_0^1 \int_0^2 (2x+2y)\, dy\, dx = \int_0^1 \left[2xy + y^2\right]_0^2 dx$$
$$= \int_0^1 (4x+4)\, dx = 6\,.$$

b) $\text{rot } \tilde{\boldsymbol{f}} = \frac{\partial \tilde{f}_2}{\partial x} - \frac{\partial \tilde{f}_1}{\partial y} = 3x^2 + 3y^2.$

For the direct calculation of the line integral of $\tilde{\boldsymbol{f}}$ over $\mathbf{c}_2$ one would have to parametrize the edges and compute line integrals over the individual edges. It's easier to use Green's theorem:

$$\int_{\partial R} \tilde{\boldsymbol{f}}(x,y)\, d(x,y) = \int_R \text{rot } \tilde{\boldsymbol{f}}(x,y)\, d(x,y) = \int_0^1 \int_0^2 3x^2 + 3y^2\, dy\, dx$$
$$= \int_0^1 \left[3x^2 y + y^3\right]_0^2 dx = \int_0^1 \left[6x^2 + 8\right] dx = \left[2x^3 + 8x\right]_0^1 = 10.$$

For $\mathbf{c}_3$ we compute:

$$\mathbf{c}_3(t) = \begin{pmatrix}1\\t^2\end{pmatrix} \qquad \dot{\mathbf{c}}_3(t) = \begin{pmatrix}0\\2t\end{pmatrix} \qquad \tilde{\boldsymbol{f}}\left(\mathbf{c}_3(t)\right) = \begin{pmatrix}\cdots\\t^4+1\end{pmatrix}$$

$$\int\limits_{\mathbf{c}_3} \tilde{\boldsymbol{f}}\,(x,y)\, d(x,y) = \int_0^3 < \tilde{\boldsymbol{f}}\,(\mathbf{c}_3(t)), \dot{\mathbf{c}}_3(t) > dt = \int_0^3 2t(t^4+1)dt = \left[\frac{t^6}{3} + t^2\right]_0^3 = 3^5 + 3^2 = 252.$$

Exercise 2) (5 + 1 + 3 + 1 Points)

Given are

$$K := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,:\, 0 \leq x^2 + y^2 + z^2 \leq 16,\, y \geq 0 \right\},$$

and the vector field

$$\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3,\ \boldsymbol{f}(x, y, z) = \begin{pmatrix} x + y^2 \\ 2y \\ 3z + x^2 \end{pmatrix}.$$

a) Compute $\displaystyle\int_K \operatorname{div} \boldsymbol{f}(x, y, z)\, d(x, y, z)$.

b) K is bounded by a flat surface W and a curved surface M. Provide the parametrization of W.

c) Compute the flow of $\boldsymbol{f}$ through W, i.e

$$\int_W \boldsymbol{f} \cdot dO.$$

d) According to a) and c) how large is the flow through the curved part of the edges of K, i.e

$$\int_M \boldsymbol{f} \cdot dO\,?$$

Solution:

a) $\operatorname{div} \boldsymbol{f}(x, y, z) = 1 + 2 + 3 = 6$.

Parametrization of K:

Spherical coordinates: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ r\sin(\varphi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$

$0 \leq r^2 = x^2 + y^2 + z^2 \leq 16 \implies r \in [0, 4], \qquad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = r\sin(\varphi)\cos(\theta) \geq 0 \implies \varphi \in [0, \pi]$

$$\int_K \operatorname{div} \boldsymbol{f}(x, y, z)\, d(x, y, z) = \int_0^4 \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 6 \cdot r^2 \cos(\theta)\, d\theta\, d\phi\, dr$$

$$= \int_0^4 \int_0^\pi 6r^2\, [\sin(\theta)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\, d\phi\, dr$$

$$= \int_0^4 12r^2\, [\varphi]_0^\pi\, dr = 4\pi \int_0^4 3r^2\, dr = 4\pi(4^3 - 0^3).$$

b) Parametrization of W: A disk with radius 4 centered at 0 and $y = 0$.

$p(r, \theta) = (\, r\cos(\theta),\, 0,\, r\sin(\theta))^T,\ \theta \in [0, 2\pi],\ 0 \leq r \leq 4$.

c) $\dfrac{\partial p}{\partial r} = \begin{pmatrix} \cos(\theta) \\ 0 \\ \sin(\theta) \end{pmatrix}, \quad \dfrac{\partial p}{\partial \theta} = \begin{pmatrix} -r\sin(\theta) \\ 0 \\ r\cos(\theta) \end{pmatrix}$

$\dfrac{\partial p}{\partial \theta} \times \dfrac{\partial p}{\partial r} = \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}.$

$< \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}, \ \boldsymbol{f}\,(p(r,\theta)) \ > = < \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}, \begin{pmatrix} \cdots \\ 0 \\ \cdots \end{pmatrix} > = \ 0\,.$

Also $\displaystyle\int_W \boldsymbol{f} \cdot dO \ = 0\,.$

d) Following Gauss theorem and using a) and b) we have

$$\int_M \boldsymbol{f} \cdot dO \ = \int_K \operatorname{div} \ \boldsymbol{f}\,(x,y,z)\,d(x,y,z) \ - \int_W \boldsymbol{f} \cdot dO \ = \ 4^4\pi\,.$$

Classes: 27.01–31.01.25