

Analysis III for Engineering Students Sheet 6 , Homework

Exercise 1:

a) Let

$$D := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \frac{y^2}{2} - 2 \leq x \leq 4 - y^2 \right\}.$$

Sketch the set D and determine the center of mass of D with uniform mass density (mass/area) $\rho = 2$.

Hint: It holds

Mass: $M = \int_D \rho(\mathbf{x}) d\mathbf{x}$

Center of mass: $X_s = \frac{1}{M} \int_D \rho(\mathbf{x}) \mathbf{x} d\mathbf{x}$ (componentwise)

b) Let $K := \{(x, y, z)^T \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, z \geq 0\}$. Compute

$$\int_K (y^2 - x^2) d(x, y, z)$$

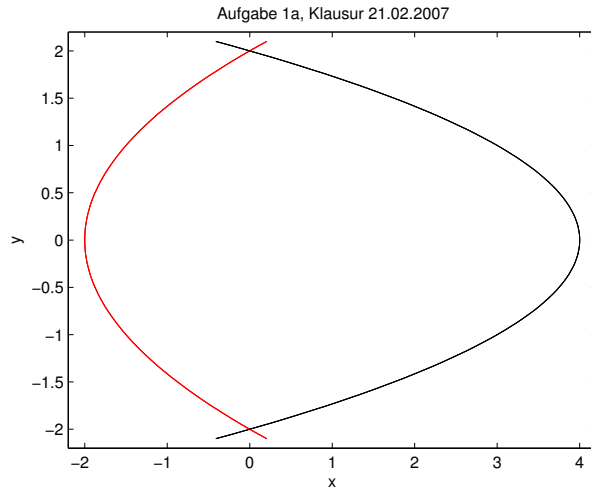
Hint:

- Using **spherical coordinates** might be helpful.
- $\cos(2t) = \cos^2(t) - \sin^2(t)$.

Solution sketch Exercise 1:

a) One can see from the sketch

$$D := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : -2 \leq y \leq 2, \frac{y^2}{2} - 2 \leq x \leq 4 - y^2 \right\}$$



To compute the center of mass we need the mass M .

$$\begin{aligned} M &= \int_{-2}^2 \int_{\frac{y^2}{2}-2}^{4-y^2} \rho \, dx \, dy = 2 \int_{-2}^2 4 - y^2 - \frac{y^2}{2} + 2 \, dy \\ &= 2 \left[-\frac{y^3}{2} + 6y \right]_{-2}^2 = 4(-4 + 12) = 32 \end{aligned}$$

Because of the symmetry, it holds for the y -component of the center of the mass: $y_s = 0$.

For the x -component of the center of the mass one obtains

$$\begin{aligned} x_s &= \frac{1}{M} \int_{-2}^2 \int_{\frac{y^2}{2}-2}^{4-y^2} \rho x \, dx \, dy = \frac{1}{M} \int_{-2}^2 2 \cdot \frac{1}{2} \left((4-y^2)^2 - \frac{(y^2-4)^2}{4} \right) dy \\ &= \frac{1}{16} \int_{-2}^2 \frac{3}{8} (4-y^2)^2 dy = \frac{3}{8 \cdot 16} \left[16y - \frac{8}{3}y^3 + \frac{y^5}{5} \right]_{-2}^2 \\ &= \frac{3}{8 \cdot 8} \left(32 - \frac{64}{3} + \frac{32}{5} \right) = \frac{3}{2} - 1 + \frac{3}{10} = \frac{4}{5}. \end{aligned}$$

- b) Using the transform to the spherical coordinates (see Work Sheet 3, Exercise 2) we have

$$\Phi : \begin{cases} \mathbb{R}^+ \times [0, 2\pi] \times [0, \frac{\pi}{2}] \rightarrow \mathbb{R}^3 \\ \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} r \cos \phi \cos \theta \\ r \sin \phi \cos \theta \\ r \sin \theta \end{pmatrix} \end{cases}$$

From the lecture and Homework Sheet 2 we already know $\det \mathbf{J} \Phi = r^2 \cos(\phi)$.

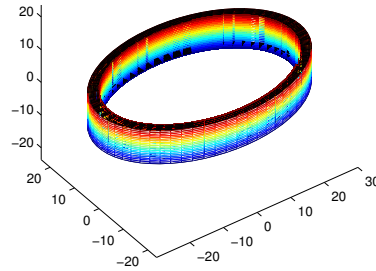
$$\begin{aligned}\int_K (y^2 - x^2) d(x, y, z) &= \\ \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} r^2 (\sin^2 \varphi - \cos^2 \varphi) \cos^2 \theta \cdot r^2 \cos \theta d\varphi d\theta dr \\ &= \int_0^1 r^4 \int_0^{\frac{\pi}{2}} \cos^3(\theta) \left(\int_0^{2\pi} -\cos 2\varphi d\varphi \right) d\theta dr = 0.\end{aligned}$$

Exercise 2:

Given is the elliptical pipe section

$$R \subset \mathbb{R}^3, \quad R: 81 \leq \left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 \leq 100, \quad -5 \leq z \leq 5.$$

with constant density ρ .



Compute the volume, mass and moment of inertia of the pipe section with respect to the y -axis using integration. Use elliptical cylindrical coordinates

$$x = 3r \cos(\varphi), \quad y = 2r \sin(\varphi), \quad z = z.$$

Hint:

$$\cos^2(\phi) = \frac{\cos(2\phi) + 1}{2}.$$

Since we do not use a calculator, there is no need to calculate the precise final value. It is sufficient to insert the integration limits into the calculated antiderivatives only.

Solution 2:

We use the transformation

$$x = 3r \cos(\varphi), \quad y = 2r \sin(\varphi), \quad z = z.$$

with parameters $r \in [9, 10]$, $\varphi \in [0, 2\pi]$, $z \in [-5, 5]$.

For the Jacobian matrix J of the coordinate transformation one calculates

$$\det \mathbf{J}(r, \varphi, z) = \det \begin{pmatrix} 3 \cos(\varphi) & -3r \sin(\varphi) & 0 \\ 2 \sin(\varphi) & 2r \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6r.$$

For the volume we get

$$V = \int_9^{10} \int_{-5}^5 \int_0^{2\pi} 6r \, d\varphi \, dz \, dr = 120\pi \int_9^{10} r \, dr = 60\pi(100 - 81) = 1140\pi.$$

For the mass it holds $M = V \cdot \rho = 1140\pi \rho$.

For the distance $a(x, y, z)$ from the point $(x, y, z)^T$ to y -Axis we have: $a(x, y, z)^2 = x^2 + z^2$.

For the moment of inertia with respect to the y -axis we calculate:

$$\begin{aligned}
 \theta_y &= \int_9^{10} \int_{-5}^5 \int_0^{2\pi} \rho(9r^2 \cos^2(\varphi) + z^2) 6r \, d\varphi \, dz \, dr \\
 &= 6\rho \int_9^{10} \int_{-5}^5 \int_0^{2\pi} 9r^3 \left(\frac{\cos(2\varphi) + 1}{2} \right) + rz^2 \, d\varphi \, dz \, dr \\
 &= 6\rho \int_9^{10} \int_{-5}^5 \left[\left(\frac{9r^3}{2} + z^2 r \right) \varphi \right]_0^{2\pi} dz \, dr = 6\rho \int_9^{10} \int_{-5}^5 9r^3 \pi + 2z^2 r \pi \, dz \, dr \\
 &= 6\rho \cdot \pi \int_9^{10} 90r^3 + 2r \left[z^3/3 \right]_{-5}^5 dr = 6\rho \cdot \pi \left(\frac{90}{4}(10^4 - 9^4) + \frac{250}{3}(100 - 81) \right) \\
 &= \rho \cdot 1.488377643527968 \cdot 10^6.
 \end{aligned}$$

Hand in until: 17.01.25