Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III for Engineering Students

Work sheet 5

Aufgabe 1: Given the following optimization problem:

$$\begin{aligned} \text{Find the minima of} \quad & f(x,y) \;=\; 1-2xy \\ \text{that satisfy the constraint} \quad & g(x,y) = x^2+4y^2-8 \le 0\,. \end{aligned} \tag{1}$$

a) Are there any local minima in the interior the admissible region, i.e. of $x^2+4y^2-8<0$? Explain your answer.
Hint: Local minima in the interior of the admissible set are also the local minima of the unconstrained problem: $\min\limits_{x,y\in\mathbb{R}} \quad f(x,y) = 1-2xy\,.$)

b) Find all global minima of f that satisfy the constraint

$$g(x,y) = \; x^2+4y^2-8 \;=\; 0$$

using the Lagrange multiplier rule. First check the regularity condition.

Remark: This exercise can also be solved by eliminating one of the variables. However, in this exercise we would like to practice the new solution method on this simple example.

c) Find all global minima of the optimization problem (1) .
Hint: use a) and b).

Solution:

a) Necessary condition for minimum in the interior:

$$\operatorname{grad} f(x,y) = (-2y,\,-2x) = (0,\,0)\,, \qquad \text{hence} \qquad x=y=0\,.$$ [**1 Point**]

Next we check the sufficient condition:

$$Hf(x,y) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \qquad \text{is indefinite} \Rightarrow \text{ no minimum here.}$$ [**1 Point**]

b) The regularity condition

$\operatorname{grad} g(x,y) = (2x,8y) \neq (0,0)^T$ is satisfied on the admissible set, since $(0,0)$ is not an admissible point. [**1 Point**]

The necessary condition for the (local) optimality is

$$\begin{aligned} \operatorname{grad} F(x,y) = \operatorname{grad}\left(f(x,y)+\lambda g(x,y)\right) = \begin{pmatrix} -2y \\ -2x \end{pmatrix}^T + \lambda \begin{pmatrix} 2x \\ 8y \end{pmatrix}^T &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}^T \\ g(x,y) = x^2+4y^2-8 \qquad\qquad\qquad &= 0\,. \end{aligned}$$

[**1 Point**]

We have to solve the system of equations

$x \cdot I: \qquad -2xy + 2\lambda x^2 = 0$

$y \cdot II: \qquad -2xy + 8\lambda y^2 = 0$

Subtracting the first equation from the second one gets

$\lambda(8y^2 - 2x^2) = 0 \implies \lambda = 0 \vee x^2 = 4y^2$.

$\lambda = 0$ corresponds to the unrestricted case and delivers the point $(0, 0)$. This point is not admissible in part b).

Hence we look for points fulfilling $x^2 = 4y^2$. Inserting this into $g(x, y) = x^2 + 4y^2 - 8 = 0$, we get

$x^2 + 4y^2 = 4y^2 + 4y^2 \overset{!}{=} 8 \implies y = \pm 1$.

and $x^2 = 4y^2 = 4 \implies x = \pm 2$.

Thus we have four candidates für local extrema:

$$P_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, P_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \quad P_4 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}.$$

[**3 points**] for finding the candidates.

f is a continuous function on the compact admissible set.

Since $f(P_1) = f(P_2) = 5$ and $f(P_3) = f(P_4) = -3$

the global minima in the admissible set are in P_3 and P_4.
The same argument is used to conclude that there must be global maxima in P_1 and P_2. They can therefore be ruled out as local minima.
[**2 points**]

Alternatively, although unnecessarily time-consuming, one could check the 2nd order sufficiency condition.

c) Since

$$\left\{(x, y)^T \in \mathbb{R}^2 \, : \, x^2 + 4y^2 - 8 \leq 0\right\}$$

is also a compact set, the continuous function f attains on it its global minimum. But not in the interior (see a)). So the global minimum of f is on the boundary. Because of b) we know that the only global Minima are in P_3 and P_4. [**1 Point**]

Exercise 2: We are looking for the extrema of the function

$$f(x,y) \;=\; 2\,\ln\left(\frac{x}{y}\right) \;+\; x \;+\; 5y$$

that fulfill the constraint

$$g(x,y) \;=\; x\,y \;-1 \;= 0.$$

a) Show that $(x_0, y_0)^T \;=\; (1,1)^T$ with a suitable fixed λ is a feasible stationary point of the Lagrangian $F \;=\; f \,+\, \lambda\, g$ and check the regularity conditions at the point $(x_0, y_0)^T \;=\; (1,1)^T$.

b) Determine the type of the stationary point $(x_0, y_0)^T \;=\; (1,1)^T$. To do so, assemble the Hessian matrix $H_{x,y}F(x_0, y_0)$ and check its definiteness on the tangent space $\ker\left(Dg(x_0, y_0)\right)$.

Solution:

a) We have $g(1,1) = 1-1 = 0$. Hence the point $(x_0, y_0)^T \;=\; (1,1)^T$ is admissible. **[1 point]**

$\nabla g(x,y) = \begin{pmatrix} y \\ x \end{pmatrix} \Longrightarrow \nabla g(1,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So the regularity condition is satisfied.
[1 point]

For f we compute

$$\nabla f(x,y) = = \begin{pmatrix} 2\,\frac{\frac{1}{y}}{\frac{x}{y}} + 1 \\ \\ 2\,\frac{\frac{-x}{y^2}}{\frac{x}{y}} + 5 \end{pmatrix} = \begin{pmatrix} 2\,\dfrac{1}{x} + 1 \\ \\ 2\,\dfrac{-1}{y} + 5 \end{pmatrix}. \qquad \textbf{[1 point]}$$

Thus, for an admissible stationary point of the Lagrange function $F = f + \lambda g$ we obtain the system of equations :

$$F_x = \frac{2}{x} + 1 + \lambda y \;=\; 0\,,$$

$$F_y = \frac{-2}{y} + 5 + \lambda x \;=\; 0\,,$$

$$g = xy - 1 = 0\,. \qquad \textbf{[1 point]}$$

For $x = y = 1$

$$F_x(1,1) : \frac{2}{1} + 1 + \lambda \;=\; 0 \Longleftrightarrow \lambda = -3\,,$$

$$F_y(1,1) = \frac{-2}{1} + 5 + \lambda \;=\; 0 \Longleftrightarrow \lambda = \;-3\,,$$

$$g(1,1) = 1 \,-\, 1 \;=\; 0\,. \qquad \textbf{[1 point]}$$

Hence $(1,1)^T$ is a stationary point of the Lagrangian with the corresponding multiplire $\lambda = -3$.

b) With $\lambda = -3$ it holds for the Hessian matrix:

$$H_{x,y}F(x,y) = \begin{pmatrix} -\frac{2}{x^2} & +\lambda \\ +\lambda & \frac{2}{y^2} \end{pmatrix} \Longrightarrow \begin{pmatrix} -2 & -3 \\ -3 & 2 \end{pmatrix} \qquad \textbf{[1 point]}$$

i.e. $H_{x,y}F(1,1)$ is indefinite ($\det H_{x,y}F(1,1) = -13$). **[1 point]**

Tangential space:
$\boldsymbol{v} = \binom{x}{y}$ with $\nabla g(1,1)^T \cdot \binom{x}{y} = 0 \Rightarrow x + y = 0$. **[1 point]**

On the tangential space:

$$(1,-1)H_{x,y}F(1,1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1,-1)\begin{pmatrix} -2 & -3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1,-1)\begin{pmatrix} 1 \\ -5 \end{pmatrix} = 6 > 0.$$

[1 point]

i.e. the Hessian matrix $H_{x,y}F(1,1)$ is positive definite on the tangential space. Hence in the point $(1,1)$ we have a strict local minimum.
[1 point]

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