Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III for Engineering Students
Sheet 5, Homework

Exercise 1:

(See lecture page 95)

For the function

$$f(x, y, z) = xy + z^2$$

find the global extrema on the intersection of the cylinder surface

$$g(x, y, z) \;=\; x^2 + y^2 - 8 \;=\; 0$$

and the plane

$$h(x, y, z) \;=\; x - y + 2z - 1 \;=\; 0.$$

Hint: First check the regularity condition.

Solution 1:

$$\begin{aligned} f(x, y, z) &= xy + z^2 \stackrel{!}{=} \min / \max \\ g(x, y, z) &= x^2 + y^2 - 8 = 0 \\ h(x, y, z) &= x - y + 2z - 1 = 0 \end{aligned}$$

Regularity condition: [2 points]

$$\text{rank} \begin{pmatrix} 2x & 2y & 0 \\ 1 & -1 & 2 \end{pmatrix} = 2$$

is satisfied $\forall \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$g(0, 0, z) = -8 \neq 0 \;\Rightarrow\;$ points with $x = y = 0$ are not admissible.
The regularity condition is satisfied on the admissible set.

We have to solve: [2 points]

$$\begin{aligned} \text{grad}(f + \lambda g + \mu h) &= 0 \\ g = h &= 0 \end{aligned}$$

$$\left.\begin{array}{rrcrcrcr} I) & y & + & \lambda \cdot 2x & + & \mu & = & 0 \\ II) & x & + & \lambda \cdot 2y & - & \mu & = & 0 \end{array}\right\} \quad y + 2\lambda x + x + 2\lambda y = 0$$

$$\begin{array}{rcrcrcl} 2z & + & \lambda \cdot 0 & + & 2\mu & = & 0 \quad \Rightarrow \quad \boxed{\mu = -z} \\ x & - & y & + & 2z & = & 1 \quad \Rightarrow \quad \boxed{z = \frac{1}{2}(1 - x + y)} \\ & & x^2 & + & y^2 & = & 8 \end{array}$$

New System : [1 point] $\mu = z = -\frac{1}{2}(1-x+y)$

$$I+II \qquad (1+2\lambda)(x+y) = 0 \Rightarrow x=-y \vee \lambda = -\frac{1}{2}$$

insert $\mu = -\frac{1}{2}(1-x+y)$ into $I)$

$$2\lambda x + y - \frac{1}{2}(1-x+y) = 0$$

$$x^2+y^2 = 8$$

Determining $P_1,\cdots P_4$ **: [3 points]**

1. case $y=-x$:

$$2\lambda x - x - \frac{1}{2}(1-x-x) = 0$$

$$x^2+x^2 = 8 \Rightarrow x^2=4 \quad x=\pm 2 \quad z=\frac{1}{2}(1-2x)$$

$$P_1 = \begin{pmatrix} 2 \\ -2 \\ -\frac{3}{2} \end{pmatrix} \quad P_2 = \begin{pmatrix} -2 \\ 2 \\ \frac{5}{2} \end{pmatrix}$$

$$f(P_1) = -4+\frac{9}{4} = -\frac{7}{4} \qquad f(P_2) = -4+\frac{25}{4} = \frac{9}{4}$$

2. case $\lambda = -\frac{1}{2}$, yet to be fulfilled: $\begin{cases} -x+y-\frac{1}{2}+\frac{x}{2}-\frac{y}{2} = 0 \\ x^2+y^2 = 8 \end{cases}$

$-x+y=1 \qquad \boxed{y=1+x}$

$z=\frac{1}{2}(1-x+y)=1 \qquad \boxed{z=1}$

$x^2+y^2 = 2x^2+2x+1=8 \;\Rightarrow\; 2x^2+2x-7=0$

$x = \frac{-1\pm\sqrt{1+14}}{2} = \frac{-1}{2}\pm\frac{\sqrt{15}}{2}\,, \qquad y=\frac{1}{2}=\frac{1}{2}\pm\frac{\sqrt{15}}{2}$

$$P_3 = \frac{1}{2}\begin{pmatrix} -1+\sqrt{15} \\ 1+\sqrt{15} \\ 2 \end{pmatrix} \qquad P_4 = \frac{1}{2}\begin{pmatrix} -1-\sqrt{15} \\ 1-\sqrt{15} \\ 2 \end{pmatrix}$$

$$\begin{aligned} f(P_3) &= \frac{1}{4}\left(\sqrt{15}-1\right)\left(\sqrt{15}+1\right)+1 \\ &= \frac{1}{4}(15-1)+1 = \frac{9}{2} \\ f(P_4) &= \frac{1}{4}\left(-\sqrt{15}+1\right)\left(-\sqrt{15}-1\right)+1 = \frac{9}{2} \end{aligned}$$

The intersection of a cylinder surface with a plane surface (not parallel to the cylinder axis) is bounded and closed $\Rightarrow$ There exists a global maximum and a global minimum. [**1 point**]

We have global maxima in P_3 and P_4 and a global minimum in P_1. [**1 point**]

Exercise 2: Given the nonlinear system of equations

$$\boldsymbol{f}(\boldsymbol{x}) := \begin{pmatrix} 4x_1^3 - 27x_1 x_2^2 + 25 \\ 4x_1^2 - 3x_2^3 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

find an approximation for a solution in the neighbourhood of $\boldsymbol{x}^{[0]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. To do this perform at least two steps of the Newton iteration starting with $\boldsymbol{x}^{[0]}$.

Solution:

Iteration: Given the point $\boldsymbol{x}^{[k]}$

- Calculate $\boldsymbol{f}(\boldsymbol{x}^{[k]})$,
- Calculate the Jacobian $\boldsymbol{J}\ \boldsymbol{f}(\boldsymbol{x}^{[k]})$,
- Solve the system $\boldsymbol{J}\ \boldsymbol{f}(\boldsymbol{x}^{[k]}) \cdot \Delta^{[k]} = -\boldsymbol{f}(\boldsymbol{x}^{[k]})$
- Let $\boldsymbol{x}^{[k+1]} = \boldsymbol{x}^{[k]} + \Delta^{[k]}$.

We have $\quad \boldsymbol{J}\ \boldsymbol{f}(x) = \begin{pmatrix} 12x_1^2 - 27x_2^2 & -54x_1 x_2 \\ 8x_1 & -9x_2^2 \end{pmatrix}$

With $\boldsymbol{x}^{[0]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we obtain:

$$\boldsymbol{J}\ \boldsymbol{f}(\boldsymbol{x}^{[0]})\Delta^{[0]} = \begin{pmatrix} -15 & -54 \\ 8 & -9 \end{pmatrix} \cdot \Delta^{[0]} = -\begin{pmatrix} 2 \\ 0 \end{pmatrix} = -\boldsymbol{f}(\boldsymbol{x}^{[0]})$$

The solution is $\Delta^{[0]} = \begin{pmatrix} \frac{2}{63} \\ \frac{16}{567} \end{pmatrix} \implies \boldsymbol{x}_1 = \begin{pmatrix} \frac{65}{63} \\ \frac{583}{567} \end{pmatrix}$

$$\boldsymbol{J}\ \boldsymbol{f}(\boldsymbol{x}_1)\Delta^{[1]} \approx \begin{pmatrix} -15.77131 & -57.28647 \\ 8.253968 & -9.515103 \end{pmatrix} \cdot \Delta^{[1]} = \begin{pmatrix} 0.05833573 \\ 0.00320282 \end{pmatrix} \approx -\boldsymbol{f}(\boldsymbol{x}^{[1]})$$

$$\boldsymbol{x}^{[2]} = \boldsymbol{x}^{[1]} + \Delta^{[1]} \approx \begin{pmatrix} 1.031149 \\ 1.027364 \end{pmatrix}, \qquad \boldsymbol{f}(\boldsymbol{x}^{[2]}) \approx \begin{pmatrix} -4.421 \times 10^{-5} \\ -5.331 \times 10^{-6} \end{pmatrix},$$

Matlab provides:

$$\begin{aligned} x_2 &= 1.031149486465732, & y_2 &= 1.027364611917862, \\ x_3 &= 1.031149301112460, & y_3 &= 1.027363890384895 \\ x_4 &= 1.031149301112562, & y_4 &= 1.027363890384492 \\ x_5 &= 1.031149301112562, & y_5 &= 1.027363890384492 \\ f_5 &= (0, 0.444089209850063 \cdot 10^{-15}) \end{aligned}$$

Hand in until: 20.12.24