Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
Prof. Dr. I. Gasser
Dr. H. P. Kiani

Analysis III for Engineering Students

Work sheet 4

Exercise 1: Determine the stationary points of the following functions and check whether they are minima, maxima or saddle points:

a) $f(\boldsymbol{x}) := \boldsymbol{x}^T \boldsymbol{A}\, \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c$ with

$$\boldsymbol{x} := \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{A} := \begin{pmatrix} 9 & -3 \\ -3 & 4 \end{pmatrix}, \quad \boldsymbol{b} := \begin{pmatrix} 6 \\ -8 \end{pmatrix}, \quad c = 24,$$

b)

$$g : \mathbb{R}^2 \to \mathbb{R}, \quad g(x,y) := x^3 + y^3 - 27xy + 25.$$

Solution 1:

a)

$$f(x,y) = (x,y)\begin{pmatrix} 9 & -3 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (6,-8)\begin{pmatrix} x \\ y \end{pmatrix} + 2024 = 9x^2 - 6xy + 4y^2 + 6x - 8y + 24.$$

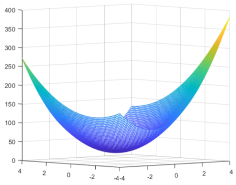
$$\begin{aligned} f_x(x,y) &= 18x - 6y + 6 = 0 \iff y - 1 = 3x. \\ f_y(x,y) &= -6x + 8y - 8 = -6x + 8(y-1) = -6x + 8\cdot 3x = 0 \implies x = 0 \\ &\implies y - 1 = 0 \implies y = 1. \end{aligned}$$

The Hessian matrix $\boldsymbol{H}\, f(x,y) = \begin{pmatrix} 18 & -6 \\ -6 & 8 \end{pmatrix}$ with
$\det(\boldsymbol{H}\,(x,y)) = 18\cdot 8 - 36 = 18 \cdot 6 > 0$ and $(\boldsymbol{H}\, f(x,y))_{11} = 18 > 0$
is positive definit. Hence we have a minimum.

Alternative classification:

$$\begin{aligned} \det(\boldsymbol{H}(x,y) - \lambda \boldsymbol{I}) = (18-\lambda)(8-\lambda) - 36 = 0 &\iff 144 - 26\lambda + \lambda^2 - 36 = 0 \\ \iff \lambda^2 - 2\cdot 13\cdot \lambda + 108 = 0 &\iff (\lambda - 13)^2 = 61. \end{aligned}$$

The eigenvalues $\lambda_{1,2} = 13 \pm \sqrt{61}$ of the Hessian are both positive. Hence we have a minimum.



b) $g(x,y) := x^3 + y^3 - 27xy + 25\,.$

$$g_x(x,y) = 3x^2 - 27y \stackrel{!}{=} 0 \iff x^2 = 9y$$
$$g_y(x,y) = 3y^2 - 27x \stackrel{!}{=} 0 \iff y^2 = 9x \iff x = \frac{y^2}{9}$$

Hence we have

$$\frac{y^4}{9^2} = 9y \iff y^4 - 9^3 y = 0 \iff y = 0 \vee y = 9\,.$$

There are two stationary points $P_1 = (0,0)$ and $P_2 = (9,9)$.

For the Hessian matrices one computes:

$$H\,g(x,y) = \begin{pmatrix} 6x & -27 \\ -27 & 6y \end{pmatrix},$$
$$H\,g(0,0) = \begin{pmatrix} 0 & -27 \\ -27 & 0 \end{pmatrix}, \quad H\,g(9,9) = \begin{pmatrix} 54 & -27 \\ -27 & 54 \end{pmatrix}.$$

In P_1 the eigenvalues of the Hessian are

$$\lambda^2 - 27^2 = 0 \implies \lambda = \pm 27$$

P_1 is a saddle point.

The following applies to the Hessian matrix at P_2

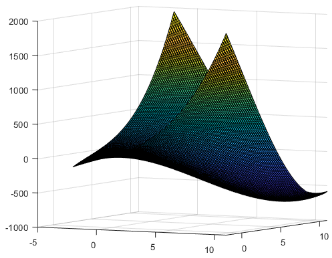
– main subdeterminants of the Hessian matrix are positive,

– alternatively: use Gerschgorins theorem

– alternatively: compute the eigenvalues:

$$(54-\lambda)^2 - 27^2 = 0 \iff 54 - \lambda = \pm 27 \implies \lambda = 54 \pm 27 > 0$$

In P_2 the function g has a (local) minimum.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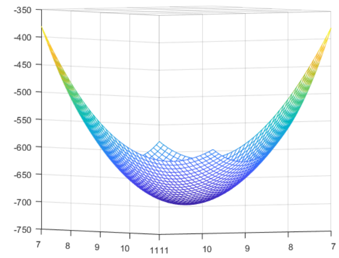
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Exercise 2:

Let $g(x,y) := y^2 \cdot x - y \cdot \exp(x+y) + 2$.

a) Show that $g(x,y) = 0$ implicitly defines a function $y(x)$ in the neighbourhood of $P_0 = (-1,1)$, i.e. the following holds locally

$$g(x,y) = 0 \Longrightarrow y = f(x), \quad f(-1) = 1\,.$$

b) Compute the first-order Taylor polynomial of f from part a) centered at $x_0 = -1$.

c) Calculate $f'(-1)$ using implicit differentiation.

d) The equation $g(x,y) = y^2 \cdot x - y \cdot \exp(x+y) + 2 = 0$ implicitly describes a curve in $\mathbb{R}^2$.

Why is it impossible for P_0 to be a singular point on the curve?

Check whether the curve has a horizontal or vertical tangent at P_0 .

Solution 2)

a) $$g(x,y) = y^2 \cdot x - y \cdot \exp(x+y) + 2 = 0$$

First we check whether the curve passes through P_0 :

$g(-1,1) = 1^2 \cdot (-1) - (1) \cdot \exp(-1+1) + 2 = -1 - 1 + 2 = 0\,.$

With $\quad g_y(x,y) = 2yx - \exp(x+y) - y \cdot \exp(x+y)$

in P_0 it holds: $\quad g_y(-1,1) = -2 - 1 - 1 = -4 \neq 0.$

The implicit function theorem says that $g(x,y)$ can be solved for y near the point $(x_0, y_0)^T := (-1,-1)^T$. This means that there exists a function $f(x)$ with $f(-1) = 1$, such that in a neighborhood of x_0 and y_0 the following equivalence holds

$$g(x,y) = 0 \Longrightarrow y = f(x), \quad f(-1) = 1\,.$$

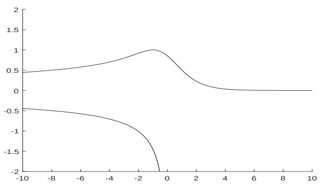
b) The theorem also states

$$f'(x) = -g_x/g_y = -\frac{y^2 - y\exp(x+y)}{2xy - \exp(x+y) - y\exp(x+y)} \Longrightarrow$$

$$f'(-1) = \frac{1 - 1\exp(0)}{-4} = 0$$

and hence

$$T_1(x;-1) = f(-1) + f'(-1)(x+1) = 1\,.$$



c) In b) one can alternatively calculate $f'(-1)$ via implicit differentiation. On the curve the following holds

$$\frac{d}{dx}\, g(x, y(x)) = \frac{d}{dx}\left(y(x)^2 \cdot x - y(x) \cdot \exp(x + y(x)) + 2\right)$$
$$= 2y(x)y'(x) \cdot x + y(x)^2 - y(x)' \cdot \exp(x + y(x)) - y(x) \cdot \exp(x + y(x)) \cdot (1 + y'(x))$$
$$= 0.$$

Inserting $x = -1$ and $y(-1) = 1$ returns:

$$-2y'(-1) + 1 - y'(-1) \cdot \exp(0) - \exp(0) \cdot (1 + y'(-1)) = -4y'(-1) = 0.$$

Hence $f'(-1) = y'(-1) = 0$.

d) Since we have shown in a) that the requirements of the implicit function theorem are fulfilled in P_0 for at least one variable, P_0 cannot be a singular point.

From part a) we already know that the curve passes through P_0 and that $g_y(-1, 1) \neq 0$.

We only need to check g_x:

$g_x(x, y) = y^2 - y \exp(x + y)$,

hence $g_x(-1, 1) = 1^2 - 1 \cdot \exp(-1 + 1) = 1 - e^0 = 0$.

The curve has a horizontal tangent in P_0 .

Classes: 02.-06.12.24