**Fachbereich Mathematik der Universität Hamburg** **WiSe 2024/25**
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# Analysis III for Engineering Students

## Sheet 4, Homework

**Exercise 1:**

The equation $g(x,y) = x^4 - x^2 + y^2 = 0$ implicitly describes a curve in $\mathbb{R}^2$.

Determine the symmetries of this curve, its singular points (+ Classification) and the curve points with horizontal or vertical tangents.

**Solution 1:**

Since $g(x,y) = g(-x,y) = g(x,-y) = g(-x,-y)$ the curve is symmetrical with respect to the $y-$ axis, the $x-$ axis and the zero point.

i) Singular Points:

$g_x(x,y) = 4x^3 - 2x = 0 \iff x = 0 \vee x = \pm\frac{1}{\sqrt{2}}$

$g_y(x,y) = 2y = 0 \iff y = 0$

Candidates: $P_0 = \binom{0}{0}$, $P_1 = \binom{\frac{1}{\sqrt{2}}}{0}$, $P_2 = \binom{-\frac{1}{\sqrt{2}}}{0}$

Since additionally $g(P_0) = 0$ holds the curve passes through $P_0$ and $P_0$ is a singular point of the curve, whereas $g(P_{1,2}) = -\frac{1}{4} \neq 0$ shows that the curve does not pass through the two other candidates.

To classify the singular point $P_0$ we calculate the second order derivatives

$g_{xx}(x,y) = 12x^2 - 2, \qquad g_{xy}(x,y) = 0, \qquad g_{yy}(x,y) = 2$

and die Hessian: $\qquad Hf(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$

Since the matrix is indefinit we have a double point.

ii) Points with horizontal tangents:

Form part i) we already know that

$g_x = 0,\ g_y \neq 0 \iff x = \pm\frac{1}{\sqrt{2}} \wedge y \neq 0\,.$

The points must lie on the curve, hence $y$ must solve:

$g(\pm\frac{1}{\sqrt{2}}, y) = -\frac{1}{4} + y^2 = 0$

and we obtain $P_{3,4} = \binom{\frac{1}{\sqrt{2}}}{\pm\frac{1}{2}}$ and $P_{5,6} = \binom{-\frac{1}{\sqrt{2}}}{\pm\frac{1}{2}}$.

iii) Points with vertical tangents:

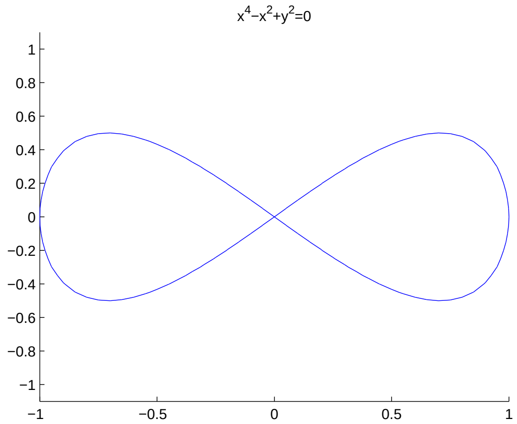
From part i) we know that $g_y = 0 \iff y = 0\,.$

We are looking for points on the curve, hence $x$ must satisfy

$x^4 - x^2 = x^2(x^2 - 1) = 0 \iff x = 0 \quad \vee \quad x = \pm 1$

The point $\binom{0}{0}$ is singular.

$P_{7,8} = \binom{\pm 1}{0}$ are points with vertical tangents.



**Exercise 2:** Given a function $g(x,y) := x^4 + y^4 + 8xy = 0\,.$

a) (i) Using the implicit function theorem show that $g(x,y)$ can be solved for $y$ near the point $(x_0,y_0)^T := (2,-2)^T$. This means that there exists a function $f(x)$ with $f(2) = -2$ , such that in some neighborhood of $x_0$ and $y_0$ the following equivalence holds

$$g(x,y) = 0 \iff y = f(x)\,.$$

(ii) Compute the first-order Taylor polynomial of the function $f$ from part (i) centered at a point $x_0 = 2\,.$

(iii) Compute the second-order Taylor polynomial of the function $f$ from part (i) centered at a point $x_0 = 2\,.$

b) Using the Implicit Function Theorem show that the solution set of

$$g(x,y,z) := (x^2 - 2e^{xy})z + 2 = 0$$

in a neighborhood of the point $P_0 := (x_0,y_0,z_0)^T := (0,1,1)^T$ can be solved for $x$. This means that there is a function $f(y,z)$ with $f(1,1) = 0$ such that in a neighborhood of $x_0, y_0, z_0$ it holds

$$g(x,y,z) = 0 \iff x = f(y,z)\,.$$

For which other variable(s) can one solve the problem using the Implicit Function Theorem?

**Solution 2:**

a) (i) $g(2,-2) = 0\,.$

$$\boldsymbol{J}\,g(x,y) = \begin{pmatrix} 4x^3 + 8y \\ 4y^3 + 8x \end{pmatrix}^T \implies \boldsymbol{J}\,g(2,-2) = \begin{pmatrix} 32 - 16 \\ -32 + 16 \end{pmatrix}^T \implies$$

One can solve both for $y$ and for $x$ near the point $(2,-2)^T$.

(ii) $T_1(x;2) = f(2) + f'(2)(x-2)$ For the first-order Taylor polynomial we also need $f'(2)$. Following the implicit function theorem we have

$$f'(x) = -g_x/g_y = -\frac{4x^3 + 8y}{4y^3 + 8x} \implies f'(2) = -\frac{16}{-16} = 1\,.$$

Alternatively : implicit differentiation

$$g(x,y(x)) = x^4 + (y(x))^4 + 8xy(x) = 0$$

$$\frac{d}{dx}g(x,y(x)) = 4x^3 + 4(y(x))^3 y'(x) + 8y(x) + 8xy'(x)$$

$$= (4x^3 + 8y(x)) + (4(y(x))^3 + 8x)y'(x) = 0$$

$$\implies y'(x) = -\frac{4x^3 + 8y}{4y^3 + 8x}$$

$$T_1(x;2) = y(2) + y'(2)(x-2) = -2 + (x-2)$$

(iii) For the second order Taylor polynomial we have to calculate $f''(2)$. We therefore differentiate

$$\frac{d}{dx}g(x,y(x)) = 4x^3 + 8y(x) + (4(y(x))^3 + 8x)y'(x) = 0$$

once again

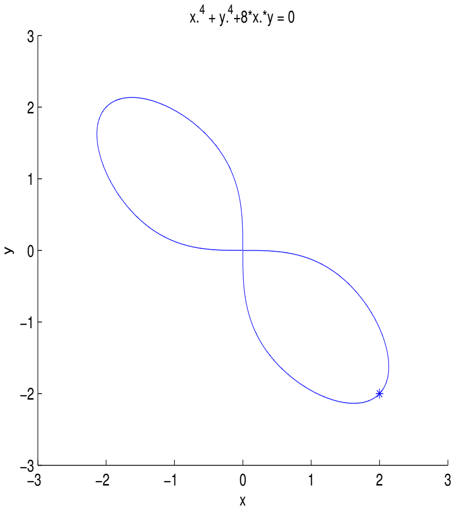
$$\frac{d^2}{dx^2}g(x,y(x)) = 12x^2 + 8y'(x) + \Big(12(y(x))^2(y'(x)) + 8\Big)\,y'(x) + \Big(4(y(x))^3 + 8x\Big)\,y''(x) = 0.$$

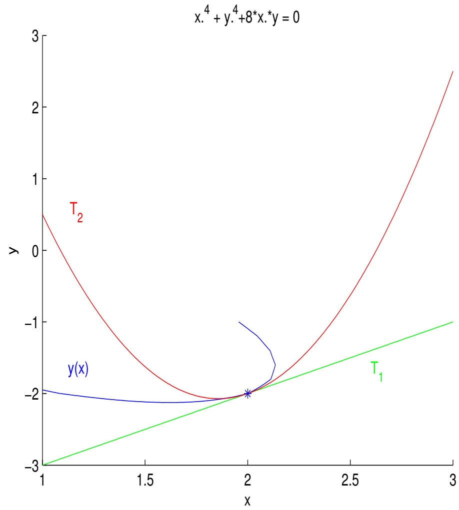
Inserting $x = 2, y(2) = -2$ and $y'(2) = 1$ gives

$$48 + 8 + 48 + 8 + (-32 + 16)y''(2) = 0 \implies y''(2) = -\frac{112}{-16} = 7\,.$$

Hence

$$T_2(x;2) = y(2) + y'(2)(x-2) + \frac{1}{2}\,y''(2)(x-2)^2 = -2 + (x-2) + \frac{7}{2}(x-2)^2$$



b) As Jacobian matrix of $g$ we have

$$\mathbf{J}g(x,y,z) = \Big((2x - 2ye^{xy})z, \quad -2xze^{xy}, \quad x^2 - 2e^{xy}\Big)$$

and hence it holds $\mathbf{J}g(0,1,1) = (-2, 0, -2)$.

Since $\frac{\partial g}{\partial x}(0,1,1) = -2$ and $\frac{\partial g}{\partial z}(0,1,1) = -2$ as $1 \times 1$-matrices are invertible, from the implicit function theorem it follows that in some neighborhood of $P_0$ there exist the functions $x(y,z)$ and $z(x,y)$ with $x(1,1) = 0$ and $g(x(y,z),y,z) = 0$ as well as $z(0,1) = 1$ and $g(x,y,z(x,y)) = 0$. The theorem does not provide the information whether it is possible to solve locally for $y$. An explicit solution of the formula for $g = 0$ to $y$ is

$y = \frac{1}{x} \cdot \ln\left(\frac{x^2}{2} + \frac{1}{z}\right)$.

This expression is not defined in any neighborhood of $(x_0, z_0) = (0,1)^T$.

**Hand in until:** 06.12.24