Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III
for Engineering Students
Work sheet 3

Exercise 1: Given a function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) := x \cdot y^2 + \cos(x+y) + 2y + 3 .$$

a) Compute the second degree Taylor polynomial T_2 of f at $(x_0, y_0) = (0,0)$.

b) Show that for the remainder $R_2(x,y) = f(x,y) - T_2(x,y)$ in the area $|x| \leq 0.2$, $|y| \leq 0.2$ the following estimate holds:

$$|f(x,y) - T_2(x,y)| \leq \frac{4}{100} .$$

c) What is the third-degree Taylor polynomial T_3 of f ar $(x_0, y_0) = (0,0)$?
Show that for all $(x,y) \in \mathbb{R}^2$, $|x| \leq 0.2$ and $|y| \leq 0.2$

$$|f(x,y) - T_3(x,y)| \leq \frac{2}{1000} .$$

holds.

Solution:

a)

$$\begin{aligned}
f(x,y) &= x \cdot y^2 + \cos(x+y) + 2y + 3 & f(0,0) &= 4 \\
f_x(x,y) &= y^2 - \sin(x+y) & f_x(0,0) &= 0 \\
f_y(x,y) &= 2xy - \sin(x+y) + 2 & f_y(0,0) &= 2 \\
f_{xx}(x,y) &= -\cos(x+y) & f_{xx}(0,0) &= -1 \\
f_{xy}(x,y) &= 2y - \cos(x+y) & f_{xy}(0,0) &= -1 \\
f_{yy}(x,y) &= 2x - \cos(x+y) & f_{yy}(0,0) &= -1
\end{aligned}$$

$$T_2(x,y) = 4 + 2y - \frac{x^2}{2} - xy - \frac{y^2}{2} .$$

b) To prove the error estimate, we calculate an upper bound for the magnitudes of all third derivatives for all $|x| \leq 0.2$ and $|y| \leq 0.2$

$$|\, f_{xxx}(x,y)\,| = |\sin(x+y)\,| \leq 1$$

$$\begin{aligned}
|\, f_{xxy}(x,y)\,| &= |\sin(x+y)\,| \leq 1 \\
|\, f_{xyy}(x,y)\,| &= |\,2 + \sin(x+y)\,| \leq 2 + 1 = 3 \\
|\, f_{yyy}(x,y)\,| &= |\sin(x+y)\,| \leq 1.
\end{aligned}$$

The magnitudes of all third derivatives of f are therefore bounded by $C := 3$. Hence, we have

$$|f(x,y) - T_2(x,y)| \le \frac{2^3}{3!} \cdot \|(x,y)\|_\infty^3 \cdot C \le \frac{8}{6} \cdot \frac{2^3}{10^3} \cdot 3 = \frac{8 \cdot 2^2}{1000} = \frac{32}{1000} < \frac{4}{100}.$$

c) Using the third derivates from part b) we get

$f_{xxx}(0,0) = f_{xxy}(0,0) = f_{yyy}(0,0) = 0, \qquad f_{xyy}(0,0) = 2.$

Hence

$$T_3(x,y) = T_2(x,y) + \frac{1}{3!} \cdot 3 f_{xyy}(0,0) x y^2 = 4 + 2y - \frac{x^2}{2} - xy - \frac{y^2}{2} + xy^2.$$

All fourth derivatives are given by $\cos(x+y)$, so that with $\tilde{C} = 1$ it follows:

$$|f(x,y) - T_3(x,y)| \le \frac{2^4}{4!} \cdot \|(x,y)\|_\infty^4 \cdot \tilde{C} \le \frac{2}{3} \cdot \frac{2^4}{10^4} = \frac{32}{30000} < \frac{33}{30 \cdot 1000} = \frac{1.1}{1000} < \frac{2}{1000}.$$

Exercise 2:

a) Describe the following subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$ in words and describe them using polar, cylindrical or spherical coordinates.

$$M_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 4 \right\},$$

$$M_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 4, \ y \geq 0 \right\},$$

$$M_3 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 4, \ x \geq 0 \right\},$$

$$M_4 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ x^2 + y^2 \leq 4, 0 \leq z \leq 5 \right\},$$

$$M_5 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ x^2 + y^2 + z^2 \leq 4 \right\},$$

$$M_6 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ x^2 + y^2 + z^2 \leq 4, \ y \geq 0 \right\},$$

$$M_7 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ x^2 + y^2 + z^2 \leq 4, \ z \geq 0 \right\}.$$

b) Describe the boundaries of the sets from a) using polar, cylindrical or spherical coordinates.

Solution 2:

a)

$$M_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : x = r\cos(\phi), \ y = r\sin(\phi); r \in [0,2], \ \phi \in [0, 2\pi) \right\},$$

circular disk, radius 2, center 0,

$$M_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : x = r\cos(\phi), \ y = r\sin(\phi); r \in [0,2], \ \phi \in [0, \pi) \right\},$$

upper half of the circular disk M_1,

$$M_3 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : x = r\cos(\phi),\ y = r\sin(\phi); r \in [0,2],\ \phi \in [-\frac{\pi}{2},, \frac{\pi}{2}] \right\},$$

right half of the circular disk M_1,

$$M_4 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : x = r\cos(\phi),\ y = r\sin(\phi); r \in [0,2],\ \phi \in [0,2\pi),\ z \in [0,5] \right\},$$

right circular Cylinder with radius 2, axis of rotation = z-axis,

base on the x-y plane and height 5,

$$M_5 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : x = r\cos(\phi)\cos(\theta), y = r\sin(\phi)\cos(\theta), z = r\sin(\theta), \right.$$

$$\left. r \in [0,2],\ \phi \in [0,2\pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \right\},$$

$$M_6 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : x = r\cos(\phi)\cos(\theta), y = r\sin(\phi)\cos(\theta), z = r\sin(\theta), \right.$$

sphere with radius 2 around zero,

$$\left. r \in [0,2],\ \phi \in [0,\pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \right\},$$

(rear) half of the sphere with radius 2 around zero with $y \geq 0$,

$$M_7 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : x = r\cos(\phi)\cos(\theta), y = r\sin(\phi)\cos(\theta), z = r\sin(\theta), \right.$$

$$\left. r \in [0,2],\ \phi \in [0,2\pi), \theta \in [0, \frac{\pi}{2}] \right\},$$

upper half of the sphere with radius 2 around zero b)

$$\partial M_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : x = 2\cos(\phi),\ y = 2\sin(\phi);\ \phi \in [0, 2\pi) \right\},$$

$$\partial M_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : x = 2\cos(\phi),\ y = 2\sin(\phi);\ \phi \in [0, \pi) \right\} \cup, \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbb{R}^2 \ : x \in [-2, 2] \right\}$$

$$\partial M_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : x = 2\cos(\phi),\ y = 2\sin(\phi); \phi \in [-\frac{\pi}{2},, \frac{\pi}{2}] \right\} \cup, \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathbb{R}^2 \ : y \in [-2, 2] \right\}$$

$$\partial M_4 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : x = 2\cos(\phi),\ y = 2\sin(\phi);\ \phi \in [0, 2\pi),\ z \in [0, 5] \right\}$$

$$\cup \left\{ (x, y, 0)^T \in \mathbb{R}^3 \ : x = r\cos(\phi),\ y = r\sin(\phi); r \in [0, 2],\ \phi \in [0, 2\pi) \right\}$$

$$\cup \left\{ (x, y, 5)^T \in \mathbb{R}^3 \ : x = r\cos(\phi),\ y = r\sin(\phi); r \in [0, 2],\ \phi \in [0, 2\pi) \right\}.$$

$$\partial M_5 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : x = 2\cos(\phi)\cos(\theta), y = 2\sin(\phi)\cos(\theta), z = 2\sin(\theta), \right.$$

$$\left. \phi \in [0, 2\pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \right\},$$

$$\partial M_6 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = 2\cos(\phi)\cos(\theta), y = 2\sin(\phi)\cos(\theta), z = 2\sin(\theta), \phi \in [0, \pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \right\}$$

$$\cup \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \in \mathbb{R}^3 \ : x = r\cos(\theta), z = r\sin(\theta), r \in [0, 2],\ \theta \in [0, 2\pi] \right\},$$

$$\partial M_7 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = 2\cos(\phi)\cos(\theta), y = 2\sin(\phi)\cos(\theta), z = 2\sin(\theta), \phi \in [0, 2\pi), \theta \in [0, \frac{\pi}{2}] \right\}$$

$$\cup \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in \mathbb{R}^3 \ : x = r\cos(\phi), y = r\sin(\phi), r \in [0, 2],\ \phi \in [0, 2\pi] \right\}.$$

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