Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
Prof. Dr. I. Gasser
Dr. H. P. Kiani

Analysis III for Engineering Students

Sheet 3, Homework

Exercise 1: (6+4 Points) Let

$$f(x,y,z) = 2 + xz + y^2 + e^x y^2 \cos(z)\,.$$

a) Compute the second degree Taylor polynomial T_2 of f at $\boldsymbol{x}_0 = (x_0, y_0, z_0)^T := (0, 1, \pi)^T$.

b) Show that for the remainder $R_2(x,y,z) = f(x,y,z) - T_2(x,y,z)$ the following estimate holds

$$|R_2(x,y,z)| \le 0.02 \qquad \forall\, \boldsymbol{x} = (x,y,z)^T \in \mathbb{R}^3 : \|\boldsymbol{x} - \boldsymbol{x}_0\|_\infty \le 0.1\,.$$

Solution:

a)

$$f(x,y,z) = 2 + xz + y^2 + e^x y^2 \cos(z), \qquad f(0,1,\pi) = 2 + 1 + \cos(\pi) = 2\,.$$

$$\begin{array}{ll}
f_x = z + e^x y^2 \cos(z), & f_x(0,1,\pi) = \pi - 1 \\
f_y = 2y + 2ye^x \cos(z), & f_y(0,1,\pi)) = 2 - 2 = 0 \\
f_z = x - e^x y^2 \sin(z), & f_z(0,1,\pi) = 0 - 0 = 0 \\
f_{xx} = e^x y^2 \cos(z), & f_{xx}(0,1,\pi) = -1 \\
f_{xy} = 2e^x y \cos(z), & f_{xy}(0,1,\pi) = -2 \\
f_{xz} = 1 - e^x y^2 \sin(z), & f_{xz}(0,1,\pi) = 1 \\
f_{yy} = 2 + 2e^x \cos(z), & f_{yy}(0,1,\pi) = 2 - 2 = 0 \\
f_{yz} = -2ye^x \sin(z), & f_{yz}(0,1,\pi) = 0 \\
f_{zz} = -e^x y^2 \cos(z), & f_{zz}(0,1,\pi) = 1
\end{array}$$

$$\begin{aligned}
T_2(x,y,z) \;&=\; f(0,1,\pi) \;+\; \operatorname{grad} f(0,1,\pi)^T \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \\
&\quad + \frac{1}{2}\,(x - x_0, y - y_0, z - z_0) Hf(0,1,\pi) \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \\
&=\; f(0,1,\pi) \;+\; f_x(0,1,\pi)(x-0) \;+\; f_y(0,1,\pi)(y-1) \;+\; f_z(0,1,\pi)(z-\pi) \\
&\quad + \frac{1}{2}\,(x, y-1, z-\pi) \begin{pmatrix} -1 & -2 & 1 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x - 0 \\ y - 1 \\ z - \pi \end{pmatrix} \\
&= 2 + \pi x - x - \frac{x^2}{2} - 2x(y-1) + x(z-\pi) + \frac{(z-\pi)^2}{2}.
\end{aligned}$$

Alternative notation

$$\begin{aligned}
T_2(x,y,z) \;&=\; f(0,1,\pi) \;+\; f_x(0,1,\pi)(x-x_0) \;+\; f_y(0,1,\pi)(y-y_0) \;+\; f_z(0,1,\pi)(z-z_0) \\
&+ \frac{1}{2!}\,\Big(f_{xx}(0,1,\pi)(x-0)^2 + 2f_{xy}(0,1,\pi)(x-0)(y-1) \\
&+ 2f_{xz}(0,1,\pi)(x-0)(z-\pi) + f_{yy}(0,1,\pi)(y-1)^2 \\
&+ 2f_{yz}(0,1,\pi)(y-1)(z-\pi) + f_{zz}(0,1,\pi)(z-\pi)^2\Big) \\
&= 2 + \pi x - x - \frac{x^2}{2} - 2x(y-1) + x(z-\pi) + \frac{(z-\pi)^2}{2}.
\end{aligned}$$

b)

$$\begin{aligned}
f_{xxx} &= e^x y^2\cos(z), & |f_{xxx}| &\le 1.1^2\cdot e^{0.1} \\
f_{xxy} &= 2ye^x\cos(z), & |f_{xxy}| &\le 2.2\cdot e^{0.1} \\
f_{xxz} &= -e^x y^2\sin(z), & |f_{xxz}| &\le 1.1^2\cdot e^{0.1} \\
f_{xyy} &= 2e^x\cos(z), & |f_{xyy}| &\le 2e^{0.1} \\
f_{xyz} &= -2e^x y\sin(z), & |f_{xyz}| &\le 2.2\cdot e^{0.1} \\
f_{xzz} &= -e^x y^2\cos(z), & |f_{xzz}| &\le 1.1^2\cdot e^{0.1} \\
f_{yyy} &= 0, & & \\
f_{yyz} &= -2e^x\sin(z), & |f_{yyz}| &\le 2e^{0.1} \\
f_{zzy} &= -2e^x y\cos(z), & |f_{yzz}| &\le 2.2\cdot e^{0.1} \\
f_{zzz} &= e^x y^2\sin(z), & |f_{zzz}| &\le 1.1^2\cdot e^{0.1}
\end{aligned}$$

An upper bound for the magnitudes of all third derivatives is e.g.
$C := 4.4 = 2.2\cdot 4^{0.5} \;>\; 2.2\cdot e^{0.5} \;>\; 2.2\cdot e^{0.1}\,.$
Hence we can prove

$$|R_2(x,y,z)| \;\le\; \frac{3^3}{3!}\cdot C\cdot \|\,\boldsymbol{x} \;-\; \boldsymbol{x}_0\|_\infty^3 \;\le\; \frac{9}{2}\cdot 4.4\cdot 0.1^3 \;=\; \frac{9\cdot 2.2}{1000} \;=\; \frac{19.8}{1000} \;<\; 0.02\,.$$

Exercise 2:

Note: To solve this problem, you do not need to calculate a single derivative exactly!

Calculate the second-degree Taylor polynomial T_2 for the function

$$f(x,y) \;=\; xy \;+\; \cos(x)\, e^y + \sin(\frac{x+y}{2})$$

at $\mathbf{x_0} \;=\; \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and show that for all

$$(x,y)^T \;\in \mathbb{R}^2 \quad \text{with} \quad |x| \;\leq\; 0.15\,,\;\; |y| \;\leq\; 0.2$$

the following estimate holds

$$|\, R_2\,(x,y;\mathbf{x_0})\,| \;:=\; |\, f(x,y) \;-\; T_2\,(x,y;\mathbf{x_0})\,| \;\leq 0.05\,.$$

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Solution 2:

The polynomial term xy is reproduced exactly.

Using the power series of $\cos, \sin, \exp$ we obtain

$$\begin{aligned} \cos(x)\, e^y &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \mp \ldots\right)\left(1 + \frac{y^1}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} \mp \ldots\right) \\ &= 1 + \frac{y^1}{1!} + \frac{y^2}{2!} - \frac{x^2}{2!} + \text{Terms of Power } \;\geq 3 \end{aligned}$$

and

$$\sin(\frac{x+y}{2}) = \frac{\frac{x+y}{2}}{1!} - \frac{(\frac{x+y}{2})^3}{3!} \pm \text{Terms of Power } \; > 3.$$

Hence

$$T_2(x,y) = \; xy \;+\; 1 + y + \frac{y^2}{2!} - \frac{x^2}{2} + \frac{x+y}{2} = 1 + xy + \frac{x}{2} + \frac{3y}{2} - \frac{x^2}{2!} + \frac{y^2}{2}. \qquad \text{[4 Points]}$$

For the error estimation, we need an upper bound for the magnitudes of the third derivatives of f. These can all be written as

$(\pm \sin(x) \text{ or } \pm\cos(x)) \cdot e^y \;-\; \frac{1}{2^3}\cos(\frac{x+y}{2})$. Without using a calculator, this gives us

$$|\text{ third derivatives }| \leq e^{0,2} + \frac{1}{2^3} < \; e^{0,5} + \frac{1}{8} < \sqrt{4} + \frac{1}{8} < 2.4 \quad \forall x \in \mathbb{R}, |y| \leq 0.2. \quad \text{[2 Points]}$$

and

$$|\, R_2\,(x,y;\mathbf{x_0})\,| \;\leq\; \frac{2^3}{3!}\,(0.2)^3 \cdot \frac{24}{10} = \frac{4\cdot 8\cdot 8}{10000} = \frac{256}{10000} \;<\; \frac{3}{100}\,. \quad \text{[1 Point]}$$

Hand in until: 22.11.24