Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III for Engineering Students

Sheet 2, Homework

Exercise 1:

Compute the Jacobian matrices for the following functions

Wherever the determinant of the Jacobian matrix does not vanish, the respective function is (locally) reversible. For which values of the variables do the determinants of the Jacobian matrices of the given functions vanish?

$$\boldsymbol{f}^{[1]} : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + y^2 \\ xy \end{pmatrix} \end{cases} \qquad \boldsymbol{f}^{[2]} : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \\ \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u - 2v \\ u \end{pmatrix} \end{cases}$$

$$\boldsymbol{f}^{[3]} = \boldsymbol{f}^{[2]} \circ \boldsymbol{f}^{[1]}$$

$$\boldsymbol{f}^{[4]} : \begin{cases} \mathbb{R}^+ \times [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^3, \quad a, b, c \in \mathbb{R}^+ \\ \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} a \cdot r \cdot \cos\phi \cos\theta \\ b \cdot r \cdot \sin\phi \cos\theta \\ c \cdot r \cdot \sin\theta \end{pmatrix} \end{cases}$$

Note for $\boldsymbol{f}^{[4]}$: For the transformation from spherical coordinates to Cartesian coordinates

$$\boldsymbol{g} : \mathbb{R} \times [0, 2\pi] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to \mathbb{R}^3, \qquad \boldsymbol{g}\begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos(\phi)\cos(\theta) \\ r\sin(\phi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$$

we know from the lecture that

$$\det(\boldsymbol{J}\, \boldsymbol{g}\,(r, \phi, \theta)) = r^2 \cos(\theta)\,.$$

Solution 1:

$$\boldsymbol{f}^{[1]}(x, y) = \begin{pmatrix} x^2 + y^2 \\ xy \end{pmatrix}$$

$$J\boldsymbol{f}^{[1]} = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix} \qquad \det\, J\boldsymbol{f}^{[1]}(x, y) = 2(x^2 - y^2).$$

The determinant of the Jacobian vanishes for $y = \pm x$.

$$\boldsymbol{f}^{[2]}(u,v) = \begin{pmatrix} u-2v \\ u \end{pmatrix}$$
$$J\boldsymbol{f}^{[2]}(u,v) = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}, \ \det J\boldsymbol{f}^{[2]}(u,v) = 2.$$

The determinant is not equal to zero for all values of u and v.

$\boldsymbol{f}^{[3]} = \boldsymbol{f}^{[2]} \circ \boldsymbol{f}^{[1]}$

$$J\boldsymbol{f}^{[3]}(x,y) = J\boldsymbol{f}^{[2]} \cdot J\boldsymbol{f}_1(x,y) = \begin{pmatrix} 2x-2y & 2y-2x \\ 2x & 2y \end{pmatrix}$$

$$\det J\boldsymbol{f}^{[3]}(x,y) = \det J\boldsymbol{f}^{[2]} \cdot \det J\boldsymbol{f}^{[1]}(x,y) = 0 \Longrightarrow$$
$$\det J\boldsymbol{f}^{[2]} = 0 \vee \det J\boldsymbol{f}^{[1]}(x,y) = 0 \Longleftrightarrow |x| = |y|$$

$$\boldsymbol{f}^{[4]}(r,\phi,\theta) = \begin{pmatrix} a\cdot r\cdot\cos\phi\cos\theta \\ b\cdot r\cdot\sin\phi\cos\theta \\ c\cdot r\cdot\sin\theta \end{pmatrix}$$

$$J\boldsymbol{f}^{[4]}(r,\phi,\theta) = \begin{pmatrix} a\cos\phi\cos\theta & -ar\sin\phi\cos\theta & -ar\cos\phi\sin\theta \\ b\sin\phi\cos\theta & br\cos\phi\cos\theta & -br\sin\phi\sin\theta \\ c\sin\theta & 0 & cr\cos\theta \end{pmatrix}$$

Direct calculation of $\det J\boldsymbol{f}^{[4]}$:

$$\det J\boldsymbol{f}^{[4]} = a\cdot b\cdot c\left(\sin\theta \begin{vmatrix} -r\sin\phi\cos\theta & -r\cos\phi\sin\theta \\ r\cos\phi\cos\theta & -r\sin\phi\sin\theta \end{vmatrix} + r\cos\theta \begin{vmatrix} \cos\phi\cos\theta & -r\sin\phi\cos\theta \\ \sin\phi\cos\theta & r\cos\phi\cos\theta \end{vmatrix}\right)$$
$$= a\cdot b\cdot c\cdot\sin\theta\left(r^2\sin\theta\cos\theta(\cos^2\phi+\sin^2\phi)\right)$$
$$+ a\cdot b\cdot c\cdot r\cos\theta\left(r\cos^2\theta(\cos^2\phi+\sin^2\phi)\right)$$
$$= a\cdot b\cdot c\cdot r^2\cos\theta(\cos^2\theta+\sin^2\theta) = a\cdot b\cdot c\cdot r^2\cos\theta,$$

Alternatively: Using the result from the lecture and the chain rule, it is much easier to get $\boldsymbol{f}^{[4]} = \boldsymbol{h}\circ\boldsymbol{g}$ where $\boldsymbol{h}(x,y,z) = (ax, by, cz)^T$ and hence

$\boldsymbol{J}\,\boldsymbol{f}^{[4]} = \boldsymbol{J}\,\boldsymbol{h}\cdot\boldsymbol{J}\,\boldsymbol{g}$ and $\det(\boldsymbol{J}\,\boldsymbol{f}^{[4]}) = \det(\boldsymbol{J}\,\boldsymbol{h})\cdot\det(\boldsymbol{J}\,\boldsymbol{g})$

Obviously, it holds

$$\boldsymbol{J}\,\boldsymbol{h} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad \text{and} \qquad \det(\boldsymbol{J}\,\boldsymbol{h}) = abc.$$

$\det(\boldsymbol{J}\,\boldsymbol{f}^{[4]})(r,\phi,\theta) = abc\cdot r^2\cos(\theta)$.

The determinant of the Jacobian matrix only vanishes for $r = 0 \notin \mathbb{R}^+$ or $\theta = \pm\frac{\pi}{2}$.

Exercise 2:

Let $f : \mathbb{R}^3 \to \mathbb{R}$ where $f(\boldsymbol{x}) := -x^2 - y^2 + 2x + z$.

a) Give an equation for the surface $N_{\boldsymbol{x}^0}$ of the function f at the point $\boldsymbol{x}^0 = (1,2,3)^{\mathrm{T}}$ and calculate the gradient of f in $\boldsymbol{x}^0$.

b) Calculate the directional derivatives $D_{\boldsymbol{w}^{[j]}} f(\boldsymbol{x}^0)$ for $j = 1,2,3$,
$\boldsymbol{v}^{[1]} = (1,1,1)^{\mathrm{T}}$, $\boldsymbol{v}^{[2]} = (1,1,0)^{\mathrm{T}}$, $\boldsymbol{v}^{[3]} = (1,0,0)^{\mathrm{T}}$
and $\boldsymbol{w}^{[j]} := \frac{\boldsymbol{v}^{[j]}}{\| \boldsymbol{v}^{[j]} \|}$. Can you decide for $j = 1,2,3$ whether $\boldsymbol{w}^{[j]}$ is an ascent or descent direction?

c) Calculate the directional derivative $D_{\tilde{\boldsymbol{v}}} f(\boldsymbol{x}^0)$ for $\tilde{\boldsymbol{v}} = 1/\sqrt{17}\,(0,-4,1)^{\mathrm{T}}$. Is this a direction of ascent or descent? Calculate the function value at the point $\boldsymbol{x}^0 + 2\sqrt{17}\,\tilde{\boldsymbol{v}}$. Doesn't this result in a contradiction? Now calculate the function value at the point $\boldsymbol{x}^0 + \frac{\sqrt{17}}{2}\,\tilde{\boldsymbol{v}}$.
Explain your results.

Solution 2:

a) For the level set it holds

$$-x^2 - y^2 + 2x + z = -1^2 - 2^2 + 2 + 3 = 0\,.$$

$$\nabla f(x,y,z) = (-2x+2, -2y, 1)^T$$
$$\nabla f(1,2,3) = (0,-4,1)^T$$

b) $D_{\boldsymbol{w}^{[1]}} f(\boldsymbol{x}^0) = \nabla f(\boldsymbol{x}^0)^T \cdot \frac{1}{\sqrt{3}}\,\boldsymbol{v}^{[1]} = \frac{1}{\sqrt{3}}(0-4+1) < 0$: Direction of descent.
$D_{\boldsymbol{w}^{[2]}} f(\boldsymbol{x}^0) = \nabla f(\boldsymbol{x}^0)^T \cdot \frac{1}{\sqrt{2}}\,\boldsymbol{v}^{[2]} = \frac{1}{\sqrt{2}}(0-4+0) < 0$: Direction of descent.
$D_{\boldsymbol{w}^{[3]}} f(\boldsymbol{x}^0) = \nabla f(\boldsymbol{x}^0)^T \cdot \boldsymbol{v}^{[3]} = 0+0+0 = 0 \Longrightarrow \boldsymbol{w}^{[3]}$ might be a downward or upward direction or a direction in which the function remains constant.

Let's take a closer look:

$$f(\boldsymbol{x}^0 + \Delta x \cdot \boldsymbol{v}^{[3]}) - f(\boldsymbol{x}^0) = -(1+\Delta x)^2 - 2^2 + 2(1+\Delta x) + 3 = -\Delta x^2 < 0$$

The function values therefore also decrease in this direction.

c)

$$D_{\tilde{\boldsymbol{v}}} f(\boldsymbol{x}^0) = \tilde{\boldsymbol{v}} \cdot \nabla f(\boldsymbol{x}^0) = \frac{1}{\sqrt{17}}\,\nabla f(\boldsymbol{x}^0) \cdot \nabla f(\boldsymbol{x}^0) = \sqrt{17}$$

$\tilde{\boldsymbol{v}}$ is a direction of ascent in $\boldsymbol{x}^0$ (it is actually the direction of the steepest ascent).

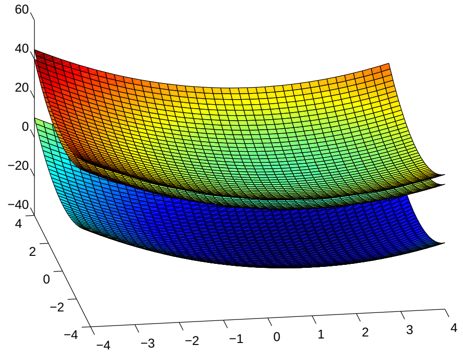
$f(\boldsymbol{x}^0 + 2\sqrt{17}\,\tilde{\boldsymbol{v}}) = f(1,-6,5) = -1 - 36 + 2 + 5 = -30\,.$

Obviously, this function value is smaller than $f(\boldsymbol{x}^0) = 0\,.$

$f(\boldsymbol{x}^0 + 0.5\sqrt{17}\,\tilde{\boldsymbol{v}}) = f(1,0,3.5) = 4.5 > f(\boldsymbol{x}^0) = 0\,.$

The reason for the seeming contradiction: Statements about ascent, descent etc. are generally only local statements.

The picture shows the level surfaces of the three points $\boldsymbol{x}^0 + 0.5\cdot\sqrt{17}\,\tilde{\boldsymbol{v}}$, $\boldsymbol{x}^0$, $\boldsymbol{x}^0 + 2\sqrt{17}\,\tilde{\boldsymbol{v}}$.



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