Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III for Engineering Students
Work sheet 1

Exercise 1:

Let

$$f : \mathbb{R}^2 \to \mathbb{R}\,, \qquad f(x,y) \;=\; \cos(2x-3y) \,+\, x^3 \,-\, y^3 \,+\, 2y^2\,.$$

a) Find all first, second and third order partial derivatives of f .

b) Determine $\operatorname{grad} f(x,y)$, $\nabla f(x,y)$ and $\Delta f(x,y)$.

Solution 1:

a)

$$\begin{aligned}
f(x,y) &= \cos(2x-3y) + x^3 - y^3 + 2y^2\,,\\
f_x(x,y) &= -2\sin(2x-3y) + 3x^2\,,\\
f_y(x,y) &= 3\sin(2x-3y) - 3y^2 + 4y\,,\\
f_{xx}(x,y) &= -4\cos(2x-3y) + 6x\,,\\
f_{xy}(x,y) &= f_{yx}(x,y) = 6\cos(2x-3y)\,,\\
f_{yy}(x,y) &= -9\cos(2x-3y) - 6y + 4\,,\\
f_{xxx}(x,y) &= 8\sin(2x-3y) + 6\,,\\
f_{xxy}(x,y) &= f_{xyx}(x,y) = f_{yxx}(x,y) = -12\sin(2x-3y)\,,\\
f_{xyy}(x,y) &= f_{yxy}(x,y) = f_{yyx}(x,y) = 18\sin(2x-3y)\,,\\
f_{yyy}(x,y) &= -27\sin(2x-3y) - 6\,.
\end{aligned}$$

b) $\operatorname{grad} f(x,y,z) = (-2\sin(2x-3y) + 3x^2,\, 3\sin(2x-3y) - 3y^2 + 4y)\,,$

$$\nabla f(x,y) = \begin{pmatrix} -2\sin(2x-3y) + 3x^2 \\ 3\sin(2x-3y) - 3y^2 + 4y \end{pmatrix},$$

$$\begin{aligned}
\Delta f(x,y) &= -4\cos(2x-3y) + 6x - 9\cos(2x-3y) - 6y + 4\\
&= -13\cos(2x-3y) + 6x - 6y + 4.
\end{aligned}$$

Exercise 2: Consider the following sets

$$M_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ x^2 + y^2 \le 1 \right\},$$

$$M_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ x^2 + y^2 < 4 \right\},$$

$$M_3 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ 1 \le x^2 + y^2 < 4 \right\},$$

$$M_4 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R},\ x^2 + y^2 \le 1 \right\},$$

$$M_5 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R},\ x^2 + y^2 + z^2 < 1 \right\},$$

$$M_6 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ (x, y) \cdot (1, 2)^T = 1 \right\},$$

$$M_7 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : \ (x, y, z) \cdot (1, 2, 1)^T < 1 \right\},$$

$$M_8 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R},\ z = x^2 + y^2 \right\}.$$

$$M_9 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ (x+3)^2 + y^2 \le 1 \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ (x-3)^2 + y^2 \le 1 \right\}.$$

a) Which are the boundary points of $M_1, \ldots, M_9$?

b) Decide for each set $M_1, \ldots, M_9$ if it is closed, open or neither closed nor open.

c) Which of the sets $M_1, \ldots, M_9$ are bounded?

d) Which sets $M_1, \ldots, M_9$ are connected? Which are convex?

Solution 2:

a)

$$\partial M_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R}, \ x^2 + y^2 = 1 \right\} \qquad \text{circle } C_1, \text{ radius 1, center 0,}$$

$$\partial M_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R}, \ x^2 + y^2 = 4 \right\} \qquad \text{circle } C_2, \text{ radius 2, center 0,}$$

$$\partial M_3 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R}, \ x^2 + y^2 \in \{1, 4\} \right\} \quad \text{two circles } C_1 \text{ and } C_2,$$

$$\partial M_4 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R}, \ x^2 + y^2 = 1 \right\} \qquad \text{right circular cylinder side,}$$

$$\partial M_5 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R}, \ x^2 + y^2 + z^2 = 1 \right\} \text{sphere, surface of a ball, radius 1, center 0,}$$

$$\partial M_6 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ (x, y) \cdot (1, 2)^T = 1 \right\} \qquad \text{line } x + 2y = 1,$$

$$\partial M_7 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : \ (x, y, z) \cdot (1, 2, 1)^T = 1 \right\} \quad \text{plane: } x + 2y + z = 1,$$

$$\partial M_8 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R}, \ z = x^2 + y^2 \right\} \qquad \text{paraboloid,}$$

$$\begin{aligned} \partial M_9 := & \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R}, \ (x+3)^2 + y^2 = 1 \right\} \\ & \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R}, \ (x-3)^2 + y^2 = 1 \right\} \text{two circles with radius 1 and centers } (\mp 3, 0)^T. \end{aligned}$$

b) M_1 : closed disc with radius 1 and center zero.

M_2 : open disc with radius 2 and center zero.

M_3 : Annulus, region between two concentric circles with centres $= 0$. The circle with radius 1 belongs to M_3 whereas the circle with radius 2 does not belong to M_3. Neither open nor closed.

M_4 : Closed infinitely long cylinder. Note: the complement is open!

M_5 : Open Ball, radius 1, centre zero.

M_6 : Line in $\mathbb{R}^2$, closed.

M_7 : Half-space in $\mathbb{R}^3$ without the dividing plane, hence open.

M_8 : Surface in $\mathbb{R}^3$, closed. The complement is open!

M_9 : Two closed discs with radius 1 and centres $(\mp 3, 0)^T$. Closed.

c) The sets M_1, M_2, M_3, M_5 and M_9 are bounded. If we choose $r \in \mathbb{R}$ large enough the sets are contained in a ball B_r with radius r and centre zero.

The sets M_4, M_6, M_7 and M_8 are unbounded. There is no $r \in \mathbb{R}$ with

$M_k \subset B_r,\ k \in \{4, 6, 7, 8\}$.

d) All sets accept M_9 are connected: Any two points belonging to one of the sets $M_k, k \neq 9$ can be connected via a curve lying in M_k. This is not true for M_9. Consider for example $(-2, 0)^T$ and $(2, 0)^T$.

Since any convex set is also connected, M_9 is not convex.

M_3 is not convex. Consider for example the line segment connecting $(-1, 0)^T$ and $(1, 0)^T$.

M_8 is not convex. Example: the line segment connecting $(-1, 0, 1)^T$ and $(1, 0, 1)^T$ does not completely belong to M_8.

All the other sets are convex.

Classes: 21.–24.11.24