Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III for Engineering Students
Homework sheet 1

Exercise 1:

a) Find all first and second order partial derivatives of

$$s(x,y,z) := xyz\,\sin(x+y+z) \qquad \text{and} \qquad g(x,y,z) := \frac{\cos^2(x)e^y}{z}\,.$$

b) Calculate for the function $f:\,\mathbb{R}^3 \longrightarrow \mathbb{R}$

$$f(x,y,z) = \arctan(x)e^y + \sin(x)\ln(1+y^2)z + x^2e^{z^2}$$

the derivative f_{xyz} as well as $\nabla f(x,y,z)$.

Solution 1:

a)

$$\begin{aligned}
s(x,y,z) :=& xyz\,\sin(x+y+z),\\
s_x(x,y,z) =& yz\,\sin(x+y+z) + xyz\cos(x+y+z)\\
s_{xx}(x,y,z) =& 2yz\,\cos(x+y+z) - xyz\sin(x+y+z)\\
s_{xy}(x,y,z) =& (z - xyz)\,\sin(x+y+z) + (xz+yz)\,\cos(x+y+z)
\end{aligned}$$

All other derivatives we get immediately due to the symmetry, since variables x,y,z are interchangeable. For example, one gets by exchanging the roles of x and z:

$$s_{zy}(x,y,z) = s_{yz}(x,y,z) = (x - xyz)\,\sin(x+y+z) + (xz+yx)\,\cos(x+y+z)$$

Or for the calculation of s_{yy} by exchanging x and y in s_{xx}

$$s_{yy}(x,y,z) = 2xz\,\cos(x+y+z) - xyz\sin(x+y+z)$$

For the calculation of f_{xz} one exchanges y and z in f_{xy}:

$$s_{xz}(x,y,z) = (y - xyz)\,\sin(x+y+z) + (xy+yz)\,\cos(x+y+z)$$

and so on.

Differentiation of $g(x, y, z) = \dfrac{\cos^2(x)e^y}{z}$ with respect to y does not change the function. Therefore:

$$\begin{aligned} g_x(x, y, z) =& \frac{-2\cos(x)\sin(x)e^y}{z} \\ g_y(x, y, z) =& g(x, y, z) = \frac{\cos^2(x)e^y}{z} \\ g_z(x, y, z) =& g_{zy}(x, y, z) = -\frac{\cos^2(x)e^y}{z^2} \, . \end{aligned}$$

$$\begin{aligned} g_{xx}(x, y, z) =& \frac{\left(-2\cos^2(x) + 2\sin^2(x)\right)e^y}{z} \, , & g_{xy}(x, y, z) =& g_x(x, y, z) \, , \\ g_{xz}(x, y, z) =& \frac{2\cos(x)\sin(x)e^y}{z^2} \, , & g_{yx}(x, y, z) =& g_x(x, y, z) \, , \\ g_{yy}(x, y, z) =& g(x, y, z) \, , & g_{yz}(x, y, z) =& g_z(x, y, z) \, , \\ g_{zx}(x, y, z) =& g_{xz}(x, y, z) \, , & g_{zy}(x, y, z) =& g_z(x, y, z) \, , \\ g_{zz}(x, y, z) =& \frac{2\cos^2(x)e^y}{z^3} \end{aligned}$$

b) In order to calculate the third order derivative f_{xyz} of

$$f(x, y, z) = \arctan(x)e^y + \sin(x)\ln(1 + y^2)z + x^2 e^{z^2}$$

it makes sense to differentiate with respect to y or z first. For example:

$$\begin{aligned} f_z(x, y, z) =& \; 0 + \sin(x)\ln(1 + y^2) + 2zx^2 e^{z^2} \\ f_{yz}(x, y, z) =& \; \sin(x)\frac{2y}{1 + y^2} + 0 \\ f_{xyz}(x, y, z) =& \; \frac{2y\cos(x)}{1 + y^2} \, . \end{aligned}$$

$$\nabla f(x, y) = \begin{pmatrix} f_x(x, y, z) \\ f_y(x, y, z) \\ f_z(x, y, z) \end{pmatrix} = \begin{pmatrix} \frac{e^y}{1+x^2} + \cos(x)\ln(1 + y^2)z + 2xe^{z^2}) \\ \arctan(x)e^y + \sin(x)\frac{2yz}{1+y^2} \\ \sin(x)\ln(1 + y^2) + 2zx^2 e^{z^2} \end{pmatrix} .$$

Exercise 2: The function

$$u(x,t) := \frac{1}{2}\left[\sin\left(\frac{2\pi}{L}(x+ct)\right) + \sin\left(\frac{2\pi}{L}(x-ct)\right)\right]$$

describes approximately the displacement of the point $x \in [0, L]$ of a vibrating string of length L at time $t > 0$

The position and the velocity of the string at time $t = 0$ are $u(x,0) = \sin\left(\frac{2\pi x}{L}\right)$ and $u_t(x,0) = 0$. These are the so-called initial values.

a) Calculate the displacement at the end points of the string, the so-called boundary values $u(0,t)$ and $u(L,t)$.

b) Show that u satisfies the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

c) Try to sketch the form of the string for $t = 0, \frac{L}{6c}, \frac{L}{4c}, \frac{L}{3c}, \frac{L}{2c}, \frac{L}{c}$.
Hint: $\sin(a+b) + \sin(a-b) = 2\sin(a)\cos(b)$.

Solution 2:

a) $u(0,t) = u(L,t) = 0$.

b) Calculate derivatives and substitute into the equation.

$$\begin{aligned}
u_x(x,t) &= \frac{1}{2}\cdot\frac{2\pi}{L}\left[\cos\left(\frac{2\pi}{L}(x+ct)\right) + \cos\left(\frac{2\pi}{L}(x-ct)\right)\right]\\
u_{xx}(x,t) &= \frac{\pi}{L}\cdot\frac{2\pi}{L}\left[-\sin\left(\frac{2\pi}{L}(x+ct)\right) - \sin\left(\frac{2\pi}{L}(x-ct)\right)\right]\\
u_t(x,t) &= \frac{1}{2}\cdot\frac{2c\pi}{L}\left[\cos\left(\frac{2\pi}{L}(x+ct)\right) - \cos\left(\frac{2\pi}{L}(x-ct)\right)\right]\\
u_{tt}(x,t) &= \frac{c\pi}{L}\cdot\frac{2c\pi}{L}\left[-\sin\left(\frac{2\pi}{L}(x+ct)\right) - \sin\left(\frac{2\pi}{L}(x-ct)\right)\right] = c^2 u_{xx}(x,t)
\end{aligned}$$

Note: In the applications, the situation is exactly the opposite: We are looking for a solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} = c^2\Delta(x), \qquad \forall\, x \in [0,L],\ t > 0$$

for given initial values, here

$$u(x,0) = \sin\left(\frac{2\pi x}{L}\right), \qquad \textit{und} \qquad u_t(x,0) = 0, \qquad \forall\, x \in [0,L]$$

and given boundary conditions, here

$$u(0,t) = u(L,t) = 0, \qquad \forall t > 0.$$

c) Using the hint $\sin(a+b)+\sin(a-b) = 2\sin(a)\cos(b)$ we get

$$u(x,t) := \frac{1}{2}\left[\sin\left(\tfrac{2\pi}{L}(x+ct)\right) + \sin\left(\tfrac{2\pi}{L}(x-ct)\right)\right] = \sin\left(\tfrac{2x\pi}{L}\right)\cdot\cos\left(\tfrac{2c\pi t}{L}\right)$$

$$u(x,0) = \sin\left(\tfrac{2\pi}{L}x\right)\cos(0) = \sin\left(\tfrac{2\pi}{L}x\right).$$

$$u(x,\tfrac{L}{6c}) = \sin\left(\tfrac{2x\pi}{L}\right)\cdot\cos\left(\tfrac{2c\pi L}{6cL}\right) = \sin\left(\tfrac{2\pi x}{L}\right)\cdot\cos(\tfrac{\pi}{3}) = \tfrac{1}{2}\,u(x,0)$$

Similarly one obtains:

$$u(x,\tfrac{L}{4c}) = \sin\left(\tfrac{2\pi x}{L}\right)\cdot\cos(\tfrac{2\pi}{L}\cdot c\cdot\tfrac{L}{4c}) = \sin\left(\tfrac{2\pi x}{L}\right)\cdot\cos(\tfrac{\pi}{2}) = 0$$

$$u(x,\tfrac{L}{3c}) = \sin\left(\tfrac{2\pi x}{L}\right)\cdot\cos(\tfrac{2\pi}{3}) = -\tfrac{1}{2}u(x,0)$$

$$u(x,\tfrac{L}{2c}) = \sin\left(\tfrac{2\pi x}{L}\right)\cdot\cos(\pi) = -u(x,0)$$

$$u(x,\tfrac{L}{c}) = \sin\left(\tfrac{2\pi x}{L}\right)\cdot\cos(2\pi) = u(x,0).$$