Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
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Analysis III for Engineering Students
Work sheet 2

Exercise 1:

a) Prove the following remark from page 24 of the lecture notes.
Remark: Let $\Phi : D \to \mathbb{R}, D \subset \mathbb{R}^3$ be a $\mathcal{C}^2$-function, then $\mathbf{curl}(\nabla\Phi) = \mathbf{0}$.
I.e. gradient fileds are curl-free everywhere.

b) Which of the following vector fields $\boldsymbol{g}$, $\boldsymbol{f}$: $\mathbb{R}^3 \to \mathbb{R}^3$,

$$\boldsymbol{g}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2+2z \\ y^2x+z \\ 2x+y \end{pmatrix} \qquad \text{und} \qquad \boldsymbol{f}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2xz \\ -2yz \\ x^2-y^2 \end{pmatrix}$$

cannot be a gradient field of a $\mathcal{C}^2$-function $\Phi, : \mathbb{R}^3 \to \mathbb{R}$?

Exercise 2: Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$f(x,y) := 3x - 5y, \qquad g(x,y) := \frac{1}{5}(x^2+y^2) + 1\,.$$

a) Calculate the gradients of f and g.

b) For f draw the contour lines (level curves)
$f^{-1}(C) := \{(x,y)^{\mathrm{T}} : f(x,y) = C\}$
for the function values $C_1 = 5, C_2 = 0$ and $C_3 = -10$.
At points $P_1 = \binom{0}{-1}, P_2 = \binom{5}{3}$ and $P_3 = \binom{-5}{-1}$ also provide the direction of the gradient.

c) For g draw the contour lines
$g^{-1}(C) := \{(x,y)^{\mathrm{T}} : g(x,y) = C\}$
for function values $C_4 = \frac{6}{5}, C_5 = \frac{21}{5}$ and $C_6 = 6$.
At points $P_4 = \binom{0}{-1}, P_5 = \binom{4}{0}$ and $P_6 = \binom{3}{4}$ also provide the direction of the gradient.

d) How does the direction of the gradient at a given point relate to the direction of the contour line through that point?

Additional task, only for the very fast students::

Given the vector field

$$\boldsymbol{f}\,(x, y, z) = (x^2 + y + 4z,\; y^2 + 2z + 5x,\; z^2 + 3x + 6y\,)^{\mathrm{T}.}$$

calculate the expressions

$$\mathrm{grad}(\mathrm{div}\;\boldsymbol{f}\,) \qquad \text{bzw.} \qquad \mathbf{rot}(\mathrm{div}\;\boldsymbol{f}\,)\,, \qquad \text{bzw.} \qquad \mathbf{rot}\,(\mathbf{rot}\;\;\boldsymbol{f}\,)$$

if these are defined. One of the expressions is equal to zero for $\boldsymbol{f}$. Using a counterexample, show that this expression does not vanish identically for any $\boldsymbol{f}$.

Classes: 04.–06.11.24