Fachbereich Mathematik der Universität Hamburg **WiSe 2024/25**
Prof. Dr. I. Gasser
Dr. H. P. Kiani

Analysis III for Engineering Students

Sheet 2, Homework

Exercise 1:

Compute the Jacobian matrices for the following functions

Wherever the determinant of the Jacobian matrix does not vanish, the respective function is (locally) reversible. For which values of the variables do the determinants of the Jacobian matrices of the given functions vanish?

$$\boldsymbol{f}^{[1]} : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + y^2 \\ xy \end{pmatrix} \end{cases} \qquad \boldsymbol{f}^{[2]} : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2 \\ \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u - 2v \\ u \end{pmatrix} \end{cases}$$

$$\boldsymbol{f}^{[3]} = \boldsymbol{f}^{[2]} \circ \boldsymbol{f}^{[1]}$$

$$\boldsymbol{f}^{[4]} : \begin{cases} \mathbb{R}^+ \times [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^3, \quad a, b, c \in \mathbb{R}^+ \\ \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} a \cdot r \cdot \cos\phi \cos\theta \\ b \cdot r \cdot \sin\phi \cos\theta \\ c \cdot r \cdot \sin\theta \end{pmatrix} \end{cases}$$

Note for $\boldsymbol{f}^{[4]}$: For the transformation from spherical coordinates to Cartesian coordinates

$$\boldsymbol{g} : \mathbb{R} \times [0, 2\pi] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to \mathbb{R}^3, \qquad \boldsymbol{g}\begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos(\phi)\cos(\theta) \\ r\sin(\phi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$$

we know from the lecture that

$$\det(\boldsymbol{J}\,\boldsymbol{g}\,(r, \phi, \theta)) = r^2 \cos(\theta).$$

Exercise 2:

Let $f : \mathbb{R}^3 \to \mathbb{R}$ where $f(\boldsymbol{x}) := -x^2 - y^2 + 2x + z$.

a) Give an equation for the surface $N_{\boldsymbol{x}^0}$ of the function f at the point $\boldsymbol{x}^0 = (1, 2, 3)^{\mathrm{T}}$ and calculate the gradient of f in $\boldsymbol{x}^0$.

b) Calculate the directional derivatives $D_{\boldsymbol{w}^{[j]}} f(\boldsymbol{x}^0)$ for $j = 1, 2, 3,$

$\boldsymbol{v}^{[1]} = (1, 1, 1)^{\mathrm{T}},\ \boldsymbol{v}^{[2]} = (1, 1, 0)^{\mathrm{T}},\ \boldsymbol{v}^{[3]} = (1, 0, 0)^{\mathrm{T}}$

and $\boldsymbol{w}^{[j]} := \frac{\boldsymbol{v}^{[j]}}{\|\boldsymbol{v}^{[j]}\|}$. Can you decide for $j = 1, 2, 3$ whether $\boldsymbol{w}^{[j]}$ is an ascent or descent direction?

c) Calculate the directional derivative $D_{\tilde{\boldsymbol{v}}} f(\boldsymbol{x}^0)$ for $\tilde{\boldsymbol{v}} = 1/\sqrt{17}\,(0, -4, 1)^{\mathrm{T}}$. Is this a direction of ascent or descent? Calculate the function value at the point $\boldsymbol{x}^0 + 2\sqrt{17}\,\tilde{\boldsymbol{v}}$. Doesn't this result in a contradiction? Now calculate the function value at the point $\boldsymbol{x}^0 + \frac{\sqrt{17}}{2}\tilde{\boldsymbol{v}}$.
Explain your results.

Hand in until: 08.11.24