Analysis III: Auditorium Exercise-06

For Engineering Students

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Let $\mathbf{f} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued and continuous function defined as

$$\begin{aligned} \mathbf{f} : D \subset \mathbb{R}^n &\to \mathbb{R}^n \\ x := (x_1, \ldots, x_n)^T &\mapsto \mathbf{f}(x_1, \ldots, x_n) \end{aligned}$$

and let $c : [a, b] \to D$, $t \mapsto c(t)$ be a piecewise C^1 curve.

Definition:

$$\int\limits_c \mathbf{f}(x) dx := \int\limits_a^b \langle \mathbf{f}(c(t)), \dot{c}(t) \rangle \; dt$$

is called **Line Integral of the Second Kind**. If the curve is **closed**, i.e., $c(a) = c(b)$, one can also write $\oint\limits_c \mathbf{f}(x) dx$.

Potential Calculation:

A vector field $\mathbf{f} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ possesses a **potential** or an **antiderivative** if there exists a C^1 function $\Phi : D \subset \mathbb{R}^n \to \mathbb{R}$ such that $\mathbf{f}$ coincides with the **gradient field** of Φ:

$$\text{grad}\ \Phi(\mathbf{x}) = \mathbf{f}(\mathbf{x})\ .$$

A C^1 vector field $\mathbf{f} : D \to \mathbb{R}^n$ possesses a potential in a simply connected region $D \subset \mathbb{R}^n$ if and only if the following integrability condition is satisfied for all $\mathbf{x} \in D$:

$$\mathbf{Jf}(\mathbf{x}) = (\mathbf{Jf}(\mathbf{x}))^T\ .$$

For $n = 2, 3$, this condition coincides with $\mathbf{rot}\ \mathbf{f}(\mathbf{x}) = \mathbf{0}$.

If there exists a potential for the vector field $\mathbf{f}$, then its called a **Conservative** field.

For a continuous vector field $\mathbf{f} : D \to \mathbb{R}^n$ with potential Φ, the following holds:

(a)
$$\int_{\mathcal{C}} \mathbf{f}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \Phi(\mathcal{C}(b)) - \Phi(\mathcal{C}(a))$$

for any piecewise C^1 curve $\mathcal{C} : [a, b] \to D$.

(b) A potential Φ associated with $\mathbf{f}$ can be calculated by

$$\Phi(\mathbf{x}) = \int_{\mathcal{C}_{\mathbf{x}}} \mathbf{f}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \text{Constant}\,.$$

Here, $\mathcal{C}_{\mathbf{x}}$ is any piecewise C^1 curve in D connecting a fixed point $\mathbf{x}_0 \in D$ to $\mathbf{x} \in D$.

Another way to calculate a potential (in addition to b) is by successively **'integrating'** the components of the vector field

$$\mathbf{f} = (f_1, f_2, f_3)^T \ ,$$

using the condition **grad** $\Phi(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, so in $\mathbb{R}^3$:

$$\begin{pmatrix} \Phi_x(x,y,z) \\ \Phi_y(x,y,z) \\ \Phi_z(x,y,z) \end{pmatrix} = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix} .$$

Consider the vector field $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathbf{f}(x, y, z) = \begin{pmatrix} 3x^2y^4z^5 + 1 \\ 4x^3y^3z^5 + 2y \\ 5x^3y^4z^4 + 3z^2 \end{pmatrix}.$$

(a) Show that $\mathbf{f}$ has a potential without explicitly calculating it.

(b) Calculate a potential by successively integrating $\mathbf{f}$.

(c) Calculate a potential using the Fundamental Theorem for Line Integrals.

(d) Along the curve $\mathcal{C} : [0, T] \to \mathbb{R}^3$ given by $\mathcal{C}(t) = (\cos t, \sin t, \sin t + \cos t)^T$ compute the curve integral $\int_{\mathcal{C}} \mathbf{f}(\mathbf{x}) \mathrm{d}\mathbf{x}$ for the cases $T = \pi$ and $T = 2\pi$.

(**a**) The space $\mathbb{R}^3$ is simply connected, and the integrability condition

$$\operatorname{curl}\mathbf{f}(x,y,z) = \begin{pmatrix} f_{3y} - f_{2z} \\ f_{1z} - f_{3x} \\ f_{2x} - f_{1y} \end{pmatrix} = \begin{pmatrix} 20x^3y^3z^4 - 20x^3y^3z^4 \\ 15x^2y^4z^4 - 15x^2y^4z^4 \\ 12x^2y^3z^5 - 12x^2y^3z^5 \end{pmatrix} = \mathbf{0}$$

is satisfied. Therefore, $\mathbf{f}(x,y,z)$ has a potential $v(x,y,z)$, i.e., $\mathbf{f} =$grad $v = (v_x, v_y, v_z)$.

(**b**) $v_x(x,y,z) = 3x^2y^4z^5 + 1 \quad \Rightarrow \quad v(x,y,z) = x^3y^4z^5 + x + c(y,z)$

$\Rightarrow \quad v_y(x,y,z) = 4x^3y^3z^5 + c_y(y,z) \stackrel{!}{=} 4x^3y^3z^5 + 2y$

$\Rightarrow \quad c_y(y,z) = 2y \quad \Rightarrow \quad c(y,z) = y^2 + k(z)$

$\Rightarrow \quad v(x,y,z) = x^3y^4z^5 + x + y^2 + k(z)$

$\Rightarrow \quad v_z(x,y,z) = 5x^3y^4z^4 + k'(z) \stackrel{!}{=} 5x^3y^4z^4 + 3z^2$

$\Rightarrow \quad k'(z) = 3z^2 \quad \Rightarrow \quad k(z) = z^3 + K \quad$ with $K \in \mathbb{R}$

$\Rightarrow \quad v(x,y,z) = x^3y^4z^5 + x + y^2 + z^3 + K$

(c) Choosing the curve $\mathcal{K}$ as the direct connection line from the point $(0,0,0)$ to the point (x,y,z), i.e., $\mathcal{K}(t) = t(x,y,z)^T$, a potential v for $\mathbf{f}$ can be calculated using the Fundamental Theorem for Line Integrals as follows:

$$\begin{aligned}
v(x,y,z) &= \int_{\mathcal{K}} \mathbf{f}(\mathbf{x})\, d\mathbf{x} + K = \int_0^1 \mathbf{f}(\mathcal{K}(t))\dot{\mathcal{K}}(t)\, dt + K \\
&= \int_0^1 \left\langle \begin{pmatrix} 3(tx)^2(ty)^4(tz)^5 + 1 \\ 4(tx)^3(ty)^3(tz)^5 + 2ty \\ 5(tx)^3(ty)^4(tz)^4 + 3(tz)^2 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle dt + K \\
&= \int_0^1 12t^{11}x^3y^4z^5 + x + 2ty^2 + 3t^2z^3\, dt + K \\
&= t^{12}x^3y^4z^5 + xt + t^2y^2 + t^3z^3 \Big|_0^1 + K \\
&= x^3y^4z^5 + x + y^2 + z^3 + K
\end{aligned}$$

(d) With $\mathcal{C}(t) = (\cos t, \sin t, \sin t + \cos t)^T$, the Fundamental Theorem for Line Integrals yields

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{f}(\mathbf{x})d\mathbf{x} &= \int_0^{\pi} \mathbf{f}(\mathcal{C}(t))\dot{\mathcal{C}}(t)\,dt = v(\mathcal{C}(\pi)) - v(\mathcal{C}(0)) \\ &= v(-1,0,-1) - v(1,0,1) = -1-1-(1+1) = -4 \end{aligned}$$

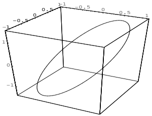

Figure: Curve $\mathcal{C}$ for $T = 2\pi$, (closed curve)

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{f}(\mathbf{x})d\mathbf{x} &= \int_0^{2\pi} \mathbf{f}(\mathcal{C}(t))\dot{\mathcal{C}}(t)\,dt = v(\mathcal{C}(2\pi)) - v(\mathcal{C}(0)) \\ &= v(1,0,1) - v(1,0,1) = 0 \end{aligned}$$

Consider a C^1 vector field $\mathbf{f} : G \to \mathbb{R}^2$ on the domain $G \subset \mathbb{R}^2$, and a compact set $D \subset G$, which is representable as a normal region with respect to both coordinate axes. Then, the following holds:

$$\int\limits_D \operatorname{rot} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \oint\limits_{\partial D} \mathbf{f}(\mathbf{x})\, d\mathbf{x}\,.$$

Here, the boundary ∂D, in the chosen parameterization for computation, must be traversed in a mathematically positive direction, i.e., counterclockwise.

Verify Green's Theorem for the vector field

$$\mathbf{f}(x, y) = (-xy - 2y, 2x + 4y^2)^T$$

and the region E enclosed by the curve $x^2 + 4y^2 = 4$.

The ellipse E can be described in Cartesian or polar coordinates as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(r,\varphi) = \begin{pmatrix} 2r\cos\varphi \\ r\sin\varphi \end{pmatrix}, \quad \begin{matrix} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \end{matrix}, \quad \Rightarrow \det \Phi(r,\varphi) = 2r$$

$$\begin{aligned} E &= \left\{(x,y)^T \in \mathbb{R}^2 \mid -2 \le x \le 2\,,\ -\sqrt{1-(x/2)^2} \le y \le \sqrt{1-(x/2)^2}\right\}, \\ Q &= \left\{(r,\varphi)^T \in \mathbb{R}^2 \mid 0 \le r \le 1\,, \quad 0 \le \varphi \le 2\pi\right\} \quad \text{with} \quad \Phi(Q) = E \end{aligned}$$

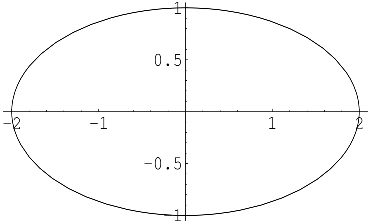

Figure: Ellipse E

Parametrization of the ellipse boundary ∂E by:

$$\mathbf{c}(\varphi) = \begin{pmatrix} 2\cos\varphi \\ \sin\varphi \end{pmatrix}, \quad 0 \le \varphi \le 2\pi$$

$$\begin{aligned}
\oint_{\partial E} \mathbf{f}(\mathbf{x})\, d\mathbf{x} &= \oint_{\mathbf{c}} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_0^{2\pi} < \mathbf{f}(\mathbf{c}(\varphi)), \dot{\mathbf{c}}(\varphi) > d\varphi \\
&= \int_0^{2\pi} \left\langle \begin{pmatrix} -2\cos\varphi\sin\varphi - 2\sin\varphi \\ 4\cos\varphi + 4\sin^2\varphi \end{pmatrix}, \begin{pmatrix} -2\sin\varphi \\ \cos\varphi \end{pmatrix} \right\rangle d\varphi \\
&= \int_0^{2\pi} 4\cos\varphi\sin^2\varphi + 4\sin^2\varphi + 4\cos^2\varphi + 4\cos\varphi\sin^2\varphi\, d\varphi \\
&= \int_0^{2\pi} 4 + 8\cos\varphi\sin^2\varphi\, d\varphi = 4\varphi + \tfrac{8}{3}\sin^3\varphi\Big|_0^{2\pi} = 8\pi
\end{aligned}$$

Solution:

$$\begin{aligned}
\int\limits_E \operatorname{rot}\mathbf{f}(\mathbf{x})\,d\mathbf{x} &= \int\limits_E (2x+4y^2)_x - (-xy-2y)_y\,d(x,y) \\
&= \int\limits_E 4 + x\,d(x,y) \;=\; \int\limits_0^1\int\limits_0^{2\pi} (4+2r\cos\varphi)2r\,d\varphi dr \\
&= 8\int\limits_0^1 r\,dr \int\limits_0^{2\pi} d\varphi + 4\int\limits_0^1 r^2\,dr \int\limits_0^{2\pi} \cos\varphi\,d\varphi \\
&= 8\pi r^2\Big|_0^1 + \frac{4r^3}{3}\Bigg|_0^1 \cdot \sin\varphi\Big|_0^{2\pi} \;=\; 8\pi
\end{aligned}$$

Green's Theorem: $\oint\limits_{\partial E} \mathbf{f}(\mathbf{x})d\mathbf{x} \;=\; 8\pi \;=\; \int\limits_E \operatorname{rot}\mathbf{f}(\mathbf{x})d\mathbf{x}$

Let $G \subset \mathbb{R}^2$ be a region and consider a C^1 mapping

$$p : G \rightarrow \mathbb{R}^3$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto p(u) = \begin{pmatrix} x(u_1, u_2) \\ y(u_1, u_2) \\ z(u_1, u_2) \end{pmatrix}.$$

If the vectors $\dfrac{\partial p(u)}{\partial u_1}$ and $\dfrac{\partial p(u)}{\partial u_2}$ are linearly independent for all $u \in G$, then,

(a) $F := \{p(u) \in \mathbb{R}^3 \mid u \in G\} = p(G)$
Surface or **Surface Patch** in $\mathbb{R}^3$.

(b) p **Parametrization** of F,

(c) G **Parameter Range** of F with respect to p,

(d) $T_F(\lambda, \mu) = p(u^0) + \lambda \dfrac{\partial p(u^0)}{\partial u_1} + \mu \dfrac{\partial p(u^0)}{\partial u_2}$ with $\lambda, \mu \in \mathbb{R}$ **Tangent Plane** at the point $p(u^0)$ to F,

(e) $\dfrac{\partial p(u^0)}{\partial u_1} \times \dfrac{\partial p(u^0)}{\partial u_2}$ **Normal Vector** to F at the point $p(u^0)$,

(f) $n(p(u)) := \dfrac{\dfrac{\partial p(u)}{\partial u_1} \times \dfrac{\partial p(u)}{\partial u_2}}{\left\| \dfrac{\partial p(u)}{\partial u_1} \times \dfrac{\partial p(u)}{\partial u_2} \right\|}$ **Unit Normal Vector** to F,

(g) $do := \left\| \dfrac{\partial p}{\partial u_1} \times \dfrac{\partial p}{\partial u_2} \right\|$ **Surface Element** and

(h) $\int_{p(G)} do := \int_G \left\| \dfrac{\partial p(u)}{\partial u_1} \times \dfrac{\partial p(u)}{\partial u_2} \right\| du$ **Surface Area** of $p(G)$.

For the surface F parametrized by the compact, measurable, and connected set D using the C^1 mapping p, i.e., $F = p(D)$, the following surface integrals are defined:

(a) **Surface Integral of the First Kind** for the continuous function $f : F \to \mathbb{R}$

$$\int_F f(\mathbf{x})\, do := \int_D f(p(\mathbf{u})) \left\| \frac{\partial p(\mathbf{u})}{\partial u_1} \times \frac{\partial p(\mathbf{u})}{\partial u_2} \right\| d\mathbf{u}\,.$$

(b) **Surface Integral of the Second Kind** for the continuous vector field $\mathbf{f} : F \to \mathbb{R}^3$

$$\begin{aligned} \int_F \mathbf{f}(\mathbf{x})\, do &:= \int_F \langle \mathbf{f}(\mathbf{x}), \mathbf{n}(\mathbf{x}) \rangle \, do \\ &= \int_D \langle \mathbf{f}(\mathbf{x}), \mathbf{n}(\mathbf{x}) \rangle \, \left\| \frac{\partial p}{\partial u_1} \times \frac{\partial p}{\partial u_2} \right\| \, d\mathbf{u} \\ &= \int_D \left\langle \mathbf{f}(p(\mathbf{u})), \frac{\partial p(\mathbf{u})}{\partial u_1} \times \frac{\partial p(\mathbf{u})}{\partial u_2} \right\rangle \, d\mathbf{u}\,. \end{aligned}$$

Note:
If the vector field $\mathbf{f}$ represents the velocity field of a stationary flow, then the surface integral $\int_F \mathbf{f}(\mathbf{x})\, do$ can be interpreted as the **flux** of $\mathbf{f}$ through the surface F, measured in the amount of fluid per unit time in the direction of the chosen normal.

Example 03

Evaluate

$$\iint_S y\,dS$$

where S is the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$.

Parameterization:

$$\vec{p}(z,\varphi) = \sqrt{3}\cos\varphi\vec{i} + \sqrt{3}\sin\varphi\vec{j} + z\vec{k}$$

The ranges of parameters:

$$0 \le z \le 6, \quad 0 \le \varphi \le 2\pi$$

The cross product:

$$\vec{p}_z \times \vec{p}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -\sqrt{3}\sin\varphi & \sqrt{3}\cos\varphi & 0 \end{vmatrix} = -\sqrt{3}\cos\varphi\vec{i} - \sqrt{3}\sin\varphi\vec{j}$$

The magnitude of this vector is,

$$\|\vec{p}_z \times \vec{p}_\varphi\| = \sqrt{3}$$

The surface integral,

$$\begin{aligned}
&\iint_S y\,dS \\
&= \iint_D \sqrt{3}\sin\varphi(\sqrt{3})\,d(z,\varphi) \\
&= 3\int_0^{2\pi}\int_0^6 \sin\varphi\,dz\,d\varphi \\
&= 3\int_0^{2\pi} 6\sin\varphi\,d\varphi = (-18\cos\varphi)\Big|_0^{2\pi} = 0
\end{aligned}$$

For the C^1 vector field $f : G \to \mathbb{R}^3$ on the domain $G \subset \mathbb{R}^3$ and the compact measurable standard region $S \subset G$, whose boundary ∂S consists of finitely many smooth surface pieces, the following holds:

$$\int_S \operatorname{div} f(x)\, dx = \oint_{\partial S} f(x)\, do\,.$$

In the calculation of the surface integral over the closed surface ∂S, hence the notation $\oint_{\partial S}$ is used, the normal vector $\dfrac{\partial p(u)}{\partial u_1} \times \dfrac{\partial p(u)}{\partial u_2}$ with respect to S points outward.

Remark:

If the vector field represents the velocity field of a stationary flow, then the surface integral $\oint_{\partial S} f(x)\, do$ can be interpreted as the flow balance through the volume S. For $\operatorname{div} f(x) = 0$ in S, according to Gauss's Divergence Theorem, $\oint_{\partial S} f(x)\, do = 0$, meaning that as much flows out of S as flows into it.

Given the solid region

$$K = \left\{ (x, y, z)^T \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 \le 9\,,\; x \le 0 \;\right\}$$

and the vector field

$$f(x, y, z) = \left(y, -x, z^3\right)^T \;.$$

(a) Sketch K.

(b) The boundary of K can be described by a planar surface piece S and a non-planar surface piece H.
Provide parametrizations for both boundary surface pieces S and H.

(c) Calculate the flux of f through both boundary surface pieces S and H.

(d) Calculate the volume integral $\int_E \operatorname{div}\,(x, y, z)\; d(x, y, z)\;.$

(a)

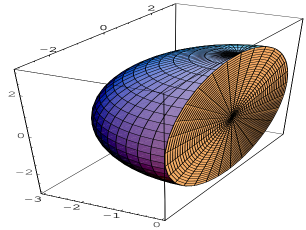

Figure: Hemisphere K

(b) Parametrization of the circular side S: $p : [0,3] \times [0, 2\pi] \to \mathbb{R}^3$ with

$$p(r,\varphi) = \begin{pmatrix} 0 \\ r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

Parametrization of the hemisphere surface H:

$q:\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}^3$ with

$$q(\varphi,\psi)=\begin{pmatrix}3\cos\varphi\cos\psi\\3\sin\varphi\cos\psi\\3\sin\psi\end{pmatrix}$$

(c) Flux through S, with the outward normal

$$\frac{\partial p}{\partial r}\times\frac{\partial p}{\partial\varphi}=\begin{vmatrix}1&2&3\\0&\cos\varphi&\sin\varphi\\0&-r\sin\varphi&r\cos\varphi\end{vmatrix}=\begin{pmatrix}r\\0\\0\end{pmatrix}$$

$$\int_S do=\int_0^3\int_0^{2\pi}\left\langle\begin{pmatrix}r\cos\varphi\\0\\r^3\sin^3\varphi\end{pmatrix},\begin{pmatrix}r\\0\\0\end{pmatrix}\right\rangle d\varphi dr=\int_0^3\int_0^{2\pi}r^2\cos\varphi d\varphi dr=0$$

Flux through H, with the outward normal

$$\frac{\partial q}{\partial \varphi} \times \frac{\partial q}{\partial \psi} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -3\sin\varphi\cos\psi & 3\cos\varphi\cos\psi & 0 \\ -3\cos\varphi\sin\psi & -3\sin\varphi\sin\psi & 3\cos\psi \end{vmatrix} = 9\cos\psi \begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ \sin\psi \end{pmatrix}$$

$$\int_H do = \int_{\pi/2}^{3\pi/2} \int_{-\pi/2}^{\pi/2} 9\cos\psi \left\langle \begin{pmatrix} 3\sin\varphi\cos\psi \\ -3\cos\varphi\cos\psi \\ 27\sin^3\psi \end{pmatrix}, \begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ \sin\psi \end{pmatrix} \right\rangle d\psi d\varphi$$

$$= \int_{\pi/2}^{3\pi/2} \int_{-\pi/2}^{\pi/2} 243\cos\psi\sin^4\psi d\psi d\varphi = 243\pi \left. \frac{\sin^5\psi}{5} \right|_{-\pi/2}^{\pi/2} = \frac{486\pi}{5}$$

(d) Using the Gaussian Integral Theorem:

$$\int_E \operatorname{div} f \, d(x,y,z) = \int_S f \, do + \int_H f \, do = \frac{486\pi}{5}$$

Alternatively: direct calculation using spherical coordinates:

$$\begin{aligned}
& \int_K \operatorname{div} f(x,y,z)\, d(x,y,z) \\
= & \int_K 3z^2 \, d(x,y,z) = \int_0^3 \int_{\pi/2}^{3\pi/2} \int_{-\pi/2}^{\pi/2} 3r^2 \sin^2\psi \cdot r^2 \cos\psi \, d\psi d\varphi dr \\
= & \int_0^3 r^4 dr \int_{\pi/2}^{3\pi/2} d\varphi \int_{-\pi/2}^{\pi/2} 3\cos\psi \sin^2\psi \, d\psi = \left.\frac{r^5}{5}\right|_0^3 \varphi\Big|_{\pi/2}^{3\pi/2} \sin^3\psi\Big|_{-\pi/2}^{\pi/2} = \frac{486\pi}{5}
\end{aligned}$$

THANK YOU