Analysis III: Auditorium Exercise-05

For Engineering Students

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Exemplary representation for a **bounded function** on a **rectangle**

$$\begin{aligned} f : \underbrace{[a,b] \times [c,d]}_{:=Q} &\rightarrow \mathbb{R} \\ (x,y) &\mapsto f(x,y)\,. \end{aligned}$$

Partition Z of the rectangle Q by

$$a = x_0 < x_1 < \cdots < x_n = b\,, \quad c = y_0 < y_1 < \cdots < y_m = d$$

into **subrectangles**

$$Q_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

with **area** $\mathrm{Area}(Q_{ij}) = (x_{i+1} - x_i) \cdot (y_{j+1} - y_j)$.

Riemann Lower Sum: (Untersumme)

$$U_f(Z) := \sum_{i=0}^{n-1} \left(\sum_{j=0}^{m-1} \inf_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}(Q_{i,j}) \right)$$

Riemann Upper Sum: (Obersumme)

$$O_f(Z) := \sum_{i=0}^{n-1} \left(\sum_{j=0}^{m-1} \sup_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}(Q_{i,j}) \right)$$

Riemann Integral: (defined only if $\sup_Z U_f(Z) = \inf_Z O_f(Z)$)

$$\int_Q f(x,y)\, d(x,y) \;:=\; \sup_Z U_f(Z) \quad \left(= \inf_Z O_f(Z) \right) .$$

Theorem: (Fubini's Theorem)

If

$$F(x) := \int_c^d f(x,y)\,dy$$

exists for all $x \in [a,b]$ and

$$G(y) := \int_a^b f(x,y)\,dx$$

exists for all $y \in [c,d]$, then

$$\int\limits_Q f(x,y)\,d(x,y) = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy\,.$$

For $Q := [0,2] \times [0,1]$, compute for the function

$$f : Q \to \mathbb{R}\,, \quad f(x,y) = 2 - x$$

(a) Compute the Riemann lower and upper sums for the following partition Z of Q

$$Q_{i,j} = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]\,,\ i,j = 1,\ldots,n$$

(b) Compute the integral of f over Q according to Fubini's theorem.

Example: 01 (a)

Riemann lower and upper sums:

$$
\begin{aligned}
U_f(Z) &= \sum_{i,j=1}^{n} \inf_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}\,(Q_{i,j}) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(2 - \frac{2i}{n}\right) \cdot \frac{2}{n^2} \right) \\
&= \frac{4}{n^2} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(1 - \frac{i}{n}\right) \right) \\
&= \frac{4}{n^2} \sum_{i=1}^{n} (n - i) \\
&= \frac{2(n^2-n)}{n^2} \;=\; 2\left(1 - \frac{1}{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
O_f(Z) &= \sum_{i,j=1}^{n} \sup_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}\,(Q_{i,j}) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(2 - \frac{2(i-1)}{n}\right) \cdot \frac{2}{n^2} \right) \;=\; 2\left(1 + \frac{1}{n}\right)
\end{aligned}
$$

The integral of f over Q according to Fubini's theorem.

$$\begin{aligned} \int_Q f(x,y)\, d(x,y) &= \int_0^1 \left(\int_0^2 2 - x \, dx \right) dy \\ &= \int_0^1 2x - \frac{x^2}{2} \Big|_0^2 \, dy \\ &= \int_0^1 2 \, dy \\ &= 2y\big|_0^1 = 2 \end{aligned}$$

Of course, one obtains:

$$2\left(1 - \frac{1}{n}\right) = U_f(Z) \le \int_Q f(x,y)\, d(x,y) = 2$$

$$\int_Q f(x,y)\, d(x,y) = 2 \le O_f(Z) = 2\left(1 + \frac{1}{n}\right) .$$

Example: 02

Compute the following integrals:

(a) $\int_{\pi}^{2\pi} \int_0^{\pi} \cos(x+y)\,dx\,dy$,

(b) $\int_R 9x^2\sqrt{y}\,d(x,y)$ with $R = [1,2] \times [1,4]$,

(c) $\int_Q \sinh z + \frac{6z^2}{(2x+y)^2}\,d(x,y,z)$ with $Q = [1,2] \times [0,1] \times [-1,1]$.

(a)

$$\begin{aligned}\int_{\pi}^{2\pi}\int_{0}^{\pi} \cos(x+y)\,dx\,dy &= \int_{\pi}^{2\pi} \sin(x+y)|_0^{\pi}\,dy \\ &= \int_{\pi}^{2\pi} \sin(\pi+y)-\sin y\,dy \\ &= (\cos y - \cos(\pi+y)|_{\pi}^{2\pi} = 4\end{aligned}$$

(b)

$R = [1,2]\times[1,4]$,

$$\begin{aligned}\int_R 9x^2\sqrt{y}\,d(x,y) &= \int_1^2\int_1^4 9x^2\sqrt{y}\,dy\,dx \\ &= \int_1^2 3x^2\left(\int_1^4 3\sqrt{y}\,dy\right)\,dx \\ &= \left(\int_1^2 3x^2\,dx\right)\cdot\left(\int_1^4 3\sqrt{y}\,dy\right) \\ &= \left(x^3\Big|_1^2\right)\cdot\left(2y^{3/2}\Big|_1^4\right) = 98\end{aligned}$$

Example: 02 (c)

$Q = [1,2] \times [0,1] \times [-1,1].$

$$
\begin{aligned}
& \int_Q \sinh z + \tfrac{6z^2}{(2x+y)^2}\, d(x,y,z) \\
= \;& \int_1^2 \int_0^1 \int_{-1}^1 \sinh z + \tfrac{6z^2}{(2x+y)^2}\, dz\, dy\, dx \\
= \;& \int_1^2 \int_0^1 \left(\cosh z + \tfrac{2z^3}{(2x+y)^2}\right)\Big|_{-1}^{1}\, dy\, dx \\
= \;& \int_1^2 \int_0^1 \tfrac{4}{(2x+y)^2}\, dy\, dx = \int_1^2 -\tfrac{4}{2x+y}\Big|_0^1\, dx \\
= \;& \int_1^2 -\tfrac{4}{2x+1} + \tfrac{2}{x}\, dx = \left(-2\ln|2x+1| + 2\ln|x|\right)\Big|_1^2 \\
= \;& -2\ln 5 + 2\ln 2 + 2\ln 3 = \ln \tfrac{36}{25}
\end{aligned}
$$

Normal regions in $\mathbb{R}^2$

A set $D \subset \mathbb{R}^2$ is called a **normal region** if

1. continuous functions $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ exist such that D has the following representation:

$$D = \{\ (x, y) \mid a \leq x \leq b\ ,\ \varphi_1(x) \leq y \leq \varphi_2(x)\ \}$$

2. continuous functions $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ exist such that D has the following representation:

$$D = \{\ (x, y) \mid \psi_1(y) \leq x \leq \psi_2(y)\ ,\ c \leq y \leq d\ \}\ .$$

A set $D \subset \mathbb{R}^3$ is called a **normal region** if continuous functions φ_1, φ_2 and ξ_1, ξ_2 exist, such that D has the following representation:

$$D = \{\, (x, y, z) \mid a \leq x \leq b\, ,\ \varphi_1(x) \leq y \leq \varphi_2(x)\, ,\ \xi_1(x, y) \leq z \leq \xi_2(x, y) \,\}$$

As in the representation in $\mathbb{R}^2$, x, y, and z can be arbitrarily interchanged.

Remark:
Often, sets D over which integration is to be performed cannot be represented by a single normal region, but only by the union of several normal regions.

Example: 03

1. 1.1 Draw the triangle D with vertices $P_1 = (-1, 1)$, $P_2 = (0, 0)$ and $P_3 = (2, 2)$ and represent it as a normal region.

   1.2 Calculate $\int_D 18y \, d(x, y)$

2. 2.1 Draw the region Z described by $x \leq 0$, $z \geq 1$, $z \leq 3$, and $x^2 + y^2 \leq 4$, and represent it as a normal region.

   2.2 Calculate $\int_Z 3x \, d(x, y, z)$

The lines through the given points are:
P_1, P_3: $g(x) = (x+4)/3$, P_1, P_2: $f_1(x) = -x$, P_2, P_3: $f_2(x) = x$.

$$D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; -1 \le x \le 2 \,,\; |x| \le y \le (x+4)/3 \right\}$$

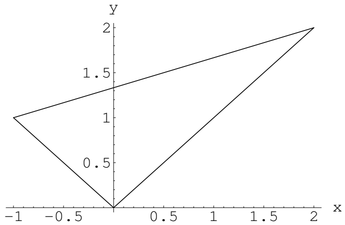


Image : Triangle D

Example: 03 (1.2)

$$\int\limits_D 18y\,d(x,y) = \int\limits_{-1}^{2}\int\limits_{|x|}^{(x+4)/3} 18y\,dy\,dx = \int\limits_{-1}^{2} 9y^2\Big|_{|x|}^{(x+4)/3}\,dx$$

$$= \int\limits_{-1}^{2} (x+4)^2 - 9x^2\,dx = \frac{(x+4)^3}{3} - 3x^3\Bigg|_{-1}^{2} = 36$$

$x \leq 0,\ z \geq 1,\ z \leq 3$ and $x^2 + y^2 \leq 4$ describes a half cylinder

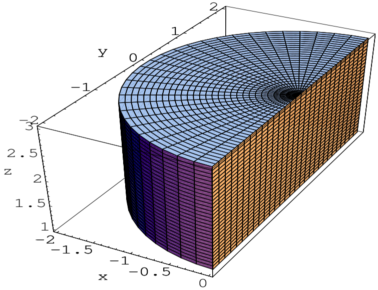


Image : Half cylinder Z

$$Z = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; -2 \leq x \leq 0\,,\ -\sqrt{4 - x^2} \leq y \leq \sqrt{4 - x^2}\,,\ 1 \leq z \leq 3 \right\}$$

Example: 03 (2.2)

$$\int_Z 3x\, d(x,y,z) = \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{1}^{3} 3x\, dz\, dy\, dx = \int_{1}^{3} dz \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3x\, dy\, dx$$

$$= 2\int_{-2}^{0} 3xy\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = 2\int_{-2}^{0} 6x\sqrt{4-x^2}\, dx$$

$$= -4\,(4-x^2)^{3/2}\Big|_{-2}^{0} = -32$$

or alternatively with transformation to cylindrical coordinates:

$$\begin{aligned}
\int_Z 3x\, d(x,y,z) &= \int_1^3 \int_{\pi/2}^{3\pi/2} \int_0^2 3r\cos(\varphi) r\, dr\, d\varphi\, dz \\
&= \int_0^2 3r^2\, dr \int_{\pi/2}^{3\pi/2} \cos(\varphi)\, d\varphi \int_1^3 dz \\
&= \left(r^3 \Big|_0^2 \right) \left(\sin(\varphi) \Big|_{\pi/2}^{3\pi/2} \right) \left(z \Big|_1^3 \right) \\
&= 8 \cdot (-2) \cdot 2 = -32
\end{aligned}$$

Consider a body $K \subset \mathbb{R}^3$ with the nonnegative continuous mass density function $\rho : K \to \mathbb{R}$.

The **mass** M of the body K is calculated by

$$M = \int_K \rho(x, y, z)\, d(x, y, z)\,.$$

The **center of mass** $\mathbf{x}_s$ of the body K is given by

$$\mathbf{x}_s = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \int_K \rho(x, y, z) \cdot x\, d(x, y, z) \\ \int_K \rho(x, y, z) \cdot y\, d(x, y, z) \\ \int_K \rho(x, y, z) \cdot z\, d(x, y, z) \end{pmatrix}.$$

The **moment of inertia** Θ_A of a body K with respect to an axis A is calculated by

$$\Theta_A = \int_K \rho(x,y,z) r^2(x,y,z)\, d(x,y,z)\,.$$

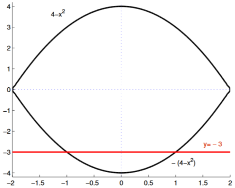
Here, $r(x,y,z)$ represents the distance of the point $(x,y,z)^T \in K$ to A.

Steiner's Theorem:
If S is an axis parallel to A and passing through the center of mass $\mathbf{x}_s$ of the body K, d is the distance of the axis A from $\mathbf{x}_s$, and M is the mass of K, then, for constant density ρ, the following holds

$$\Theta_A = Md^2 + \Theta_S.$$

Given: The set

$$D := \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2,\ y^2 \leq (4 - x^2)^2,\ y \geq -3\}.$$



- Write D as a union of normal domains (with respect to x).
- Compute the area of D.
- Compute the center of mass (centroid) of D assuming a homogeneous density $\rho = 3$.

1) Decomposition into normal domains.
We split D into three subregions:

$$D = D_1 \cup D_2 \cup D_3,$$

where

$$D_1 : -2 \leq x \leq -1, \quad x^2 - 4 \leq y \leq 4 - x^2,$$

$$D_2 : -1 \leq x \leq 1, \quad -3 \leq y \leq 4 - x^2,$$

$$D_3 : 1 \leq x \leq 2, \quad x^2 - 4 \leq y \leq 4 - x^2.$$

2) Area of D.
We denote the area by F. Notice that, by symmetry, we can double the integrals over $x \geq 0$ and include the contribution from $x \leq 0$. Hence,

$$F = 2\Bigg[\underbrace{\int_0^1 \int_{-3}^{4-x^2} 1\,dy\,dx}_{\text{region } D_2 \text{ for } 0\le x\le 1} + \underbrace{\int_1^2 \int_{x^2-4}^{4-x^2} 1\,dy\,dx}_{\text{region } D_3}\Bigg].$$

(a) For $0 \le x \le 1$:

$$\int_{-3}^{4-x^2} 1\,dy = (4 - x^2) - (-3) = 7 - x^2.$$

Hence,

$$\int_0^1 \left[(7 - x^2)\right] dx = \left[7x - \frac{x^3}{3}\right]_0^1 = 7 - \frac{1}{3} = \frac{20}{3}.$$

(b) For $1 \le x \le 2$:

$$\int_{x^2-4}^{4-x^2} 1\,dy = (4-x^2)-(x^2-4) = 8-2x^2.$$

Thus,

$$\int_1^2 (8-2x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_1^2 = \left(16-\frac{16}{3}\right) - \left(8-\frac{2}{3}\right) = \frac{10}{3}.$$

Putting these together and multiplying by 2 (due to the symmetry about $x=0$):

$$F = 2\left(\int_0^1 (7-x^2)\,dx + \int_1^2 (8-2x^2)\,dx\right) = 2\left(\tfrac{20}{3}+\tfrac{10}{3}\right) = 2\times\tfrac{30}{3} = 2\times 10 = 20.$$

3) Center of mass (centroid).

Mass of D:

Since the density is $\rho = 3$ (constant),

$$M = \int_D \rho \, d(x, y) = \rho \cdot \mathrm{Area}(D) = 3 \times 20 = 60.$$

Let (x_s, y_s) be the centroid. Then

$$x_s = \frac{1}{M} \int_D x \, \rho(x, y) \, d(x, y).$$

By symmetry (the domain is symmetric about the y-axis), we immediately get

$$x_s = 0.$$

For y_s, we write

$$y_s = \frac{1}{M}\int_D y\,\rho(x,y)\,d(x,y) = \frac{\rho}{\rho F}\int_D y\,d(x,y) = \frac{1}{F}\int_D y\,d(x,y).$$

Moreover, by symmetry arguments, the contribution to $\int y\,d(x,y)$ from D_1 and D_3 (the left and right "caps") is zero. Thus we only need to integrate over D_2:

$$D_2 : -1 \leq x \leq 1, \quad -3 \leq y \leq 4 - x^2.$$

Hence

$$y_s = \frac{1}{F}\int_{x=-1}^{1}\int_{y=-3}^{4-x^2} y\,dy\,dx.$$

Example: 04

Compute the inner integral:

$$\int_{-3}^{4-x^2} y\,dy = \frac{y^2}{2}\bigg|_{y=-3}^{y=4-x^2} = \frac{(4-x^2)^2}{2} - \frac{9}{2} = \frac{7-8x^2+x^4}{2}.$$

$$\int_{x=-1}^{1}\int_{y=-3}^{4-x^2} y\,dy\,dx = \int_{-1}^{1}\frac{7-8x^2+x^4}{2}\,dx = \frac{1}{2}\int_{-1}^{1}\left(7-8x^2+x^4\right)dx.$$

Because the integrand $7-8x^2+x^4$ is an even function, we can write

$$\int_{-1}^{1}\left(7-8x^2+x^4\right)dx = 2\int_{0}^{1}\left(7-8x^2+x^4\right)dx.$$

Hence

$$\frac{1}{2}\int_{-1}^{1}\left(7-8x^2+x^4\right)dx = \int_{0}^{1}\left(7-8x^2+x^4\right)dx.$$

We compute

$$\int_0^1 (7 - 8x^2 + x^4)\, dx = \left[7x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^1 = 7 - \frac{8}{3} + \frac{1}{5} = \frac{68}{15}.$$

Thus

$$\int_{x=-1}^{1} \int_{y=-3}^{4-x^2} y\, dy\, dx = \frac{68}{15}.$$

Therefore,

$$y_s = \frac{1}{F}\frac{68}{15} = \frac{68}{15 \cdot 20} = \frac{68}{300} = \frac{17}{75}.$$

Hence the centroid of D is

$$(x_s, y_s) = \left(0, \frac{17}{75}\right).$$

1. **Polar Coordinates:** $0 \leq r \leq R\,, \quad 0 \leq \varphi \leq 2\pi$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \qquad (\Rightarrow \det(J\Phi(r,\varphi)) = r)$$

2. **Cylindrical Coordinates:**
   $0 \leq r \leq R\,, \quad 0 \leq \varphi \leq 2\pi\,, \quad a \leq z \leq b$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r,\varphi,z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad (\Rightarrow \det(J\Phi(r,\varphi,z)) = r)$$

3. **Spherical Coordinates:**
   $0 \leq r \leq R\,, \quad 0 \leq \varphi \leq 2\pi\,, \quad -\pi/2 \leq \theta \leq \pi/2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r,\varphi,\theta) = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ r\sin\theta \end{pmatrix} \qquad (\Rightarrow \det(J\Phi(r,\varphi,\theta)) = r^2\cos\theta$$

Transformation Theorem

For continuous functions $f : K \subset \mathbb{R}^n \to \mathbb{R}$, the following holds

$$\int_K f(x)\,dx = \int_D f(\Phi(u)) \cdot |\det(J\Phi(u))|\,du$$

$D \subset \mathbb{R}^n$ compact and measurable, $K = \Phi(D)$, and the C^1 coordinate transformation : $D \to \mathbb{R}^n$.
The transformation Φ must be invertible on D^0.

Example: 05

1. Draw the quarter sphere K given by $y \leq 0$, $z \leq 0$, and $x^2 + y^2 + z^2 \leq 16$. Calculate its center of mass using the density function $\rho(x, y, z) = x^2 + y^2 + z^2 + 1$ and using spherical coordinates.
2. P is described by $x^2 + y^2 + z^2 \leq 9$, there is a sphere K with constant density ρ.
   2.1 Draw K .
   2.2 Calculate the mass and the moment of inertia of K with respect to the z-axis.
   2.3 Calculate the moment of inertia of K with respect to the axis D parallel to the z-axis and passing through the point $(2, 1, 3)^T$.

Example: 05 (1)

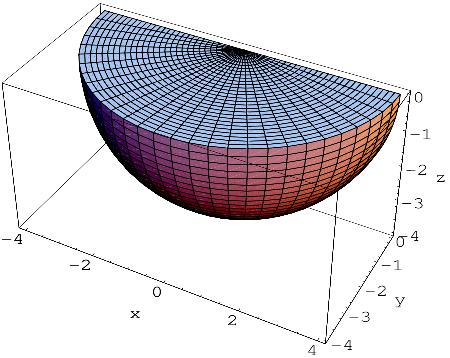


Figure: Quarter-sphere K

Example: 05 (1)

Spherical coordinates for K: $\quad 0 \leq r \leq 4\,,\ \pi \leq \varphi \leq 2\pi\,,\ -\pi/2 \leq \theta \leq 0$
with

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ r\sin(\varphi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix} = \Phi(r,\varphi,\theta), \quad \det J\Phi(r,\varphi,\theta) = r^2\cos(\theta)$$

Example: 05 (1)

Calculation of the mass M in spherical coordinates using the transformation theorem with $\rho(x, y, z) = x^2 + y^2 + z^2 + 1$:

$$\begin{aligned}
M &= \int_K (x^2 + y^2 + z^2 + 1)\, d(x, y, z) \\
&= \int_0^4 \int_\pi^{2\pi} \int_{-\pi/2}^{0} (r^2 + 1) r^2 \cos(\theta)\, d\theta\, d\varphi\, dr \\
&= \int_0^4 \int_\pi^{2\pi} r^4 + r^2\, d\varphi\, dr \\
&= \int_0^4 \pi (r^4 + r^2)\, dr = \left.\frac{(3 \cdot r^5 + 5 \cdot r^3)\pi}{15}\right|_0^4 = \frac{3392\pi}{15}
\end{aligned}$$

Example: 05 (1)

Calculation of the coordinates of the center of mass (x_s, y_s, z_s):

$$
\begin{aligned}
x_s &= \frac{1}{M} \int_K (x^2 + y^2 + z^2 + 1)x \, d(x, y, z) \\
&= \frac{1}{M} \int_0^4 \int_\pi^{2\pi} \int_{-\pi/2}^{0} (r^2 + 1) r \cos(\varphi) \cos(\theta) r^2 \cos(\theta) \, d\theta \, d\varphi \, dr \\
&= \frac{1}{M} \int_0^4 \int_\pi^{2\pi} (r^5 + r^3) \cos(\varphi) \left. \frac{\theta + \sin(\theta)\cos(\theta)}{2} \right|_{-\pi/2}^{0} d\varphi \, dr \\
&= \frac{\pi}{4M} \int_0^4 (r^5 + r^3) \left. \sin(\varphi) \right|_\pi^{2\pi} \, dr = 0
\end{aligned}
$$

This result also arises due to symmetry.

Example: 05 (1)

$$
\begin{aligned}
y_s &= \frac{1}{M}\int\limits_K (x^2+y^2+z^2+1)y\;d(x,y,z) \\
&= \frac{1}{M}\int\limits_0^4\int\limits_\pi^{2\pi}\int\limits_{-\pi/2}^{0}(r^2+1)r\sin(\varphi)\cos(\theta)r^2\cos(\theta)\;d\theta\;d\varphi\;dr \\
&= \frac{1}{M}\int\limits_0^4\int\limits_\pi^{2\pi}(r^5+r^3)\sin(\varphi)\;\left.\frac{\theta+\sin(\theta)\cos(\theta)}{2}\right|_{-\pi/2}^{0}\;d\varphi\;dr \\
&= \frac{\pi}{4M}\int\limits_0^4 (r^5+r^3)\;\cos(\varphi)|_\pi^{2\pi}\;dr = -\frac{\pi(2\cdot r^6+3\cdot r^4)\big|_0^4}{24M} \\
&= -\frac{1120\pi}{3M} = -\frac{175}{106}
\end{aligned}
$$

$$
\begin{aligned}
z_s &= \frac{1}{M}\int\limits_K (x^2+y^2+z^2+1)z\, d(x,y,z) \\
&= \frac{1}{M}\int\limits_0^4\int\limits_\pi^{2\pi}\int\limits_{-\pi/2}^{0} (r^2+1)r\sin(\theta)r^2\cos(\theta)\, d\theta\, d\varphi\, dr \\
&= \frac{1}{M}\int\limits_0^4\int\limits_\pi^{2\pi}(r^5+r^3)\left.\frac{\sin^2(\theta)}{2}\right|_{-\pi/2}^{0} d\varphi\, dr \\
&= -\frac{1}{2M}\int\limits_0^4 (r^5+r^3)\left.\varphi\right|_\pi^{2\pi}\, dr = -\frac{\pi(2\cdot r^6+3\cdot r^4)\big|_0^4}{24M} \\
&= -\frac{1120\pi}{3M} = -\frac{175}{106}
\end{aligned}
$$

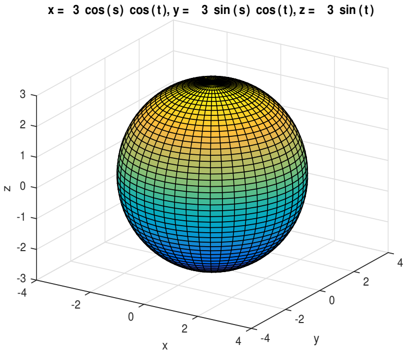


Figure: Sphere K with radius $R = 3$

Example: 05 (2.2)

Calculation of the mass M in spherical coordinates using the transformation theorem with constant density ρ:

$$\begin{aligned}
M &= \int_K \rho \, d(x,y,z) = \rho \int_0^3 \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} r^2 \cos(\theta) \, d\theta \, d\varphi \, dr \\
&= \rho \int_0^3 r^2 \, dr \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} \cos(\theta) \, d\theta \; = \; \rho \left(\frac{r^3}{3} \right) \Big|_0^3 \, (\varphi)|_0^{2\pi} \, (\sin(\theta))|_{-\pi/2}^{\pi/2} \\
&= \rho \frac{3^3}{3} \cdot 2\pi \cdot 2 \; = \; \rho \frac{4\pi 3^3}{3} \; = \; 36\pi\rho
\end{aligned}$$

Example: 05 (2.2)

Calculation of the moment of inertia with respect to the z-axis in spherical coordinates using the transformation theorem with constant density ρ and the addition theorem. $\cos^3(\theta) = (3\cos(\theta) + \cos(3\theta))/4$

$$
\begin{aligned}
\Theta_z &= \int_K \rho(x^2+y^2)\, d(x,y,z) \\
&= \rho \int_0^3 \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left(r^2\cos^2(\varphi)\cos^2(\theta) + r^2\sin^2(\varphi)\cos^2(\theta)\right) r^2\cos(\theta)\, d\theta\, d\varphi\, dr \\
&= \rho \int_0^3 r^4 dr \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} \cos^3(\theta) d\theta \\
&= \rho \left(\frac{r^5}{5}\right)\Bigg|_0^3 (\varphi)|_0^{2\pi}\, \frac{1}{4}\left(3\sin(\theta) + \frac{1}{3}\sin(3\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= \rho\frac{3^5}{5} \cdot 2\pi \cdot \frac{4}{3} = \frac{648\pi\rho}{5}
\end{aligned}
$$

Example: 05 (2.3)

Since the center of mass of P is at the origin due to symmetry reasons, according to the Steiner's theorem,

$$\begin{aligned}\Theta_D &= Md^2 + \Theta_{z\text{-axis}} \\ &= 36\pi\rho(2^2 + 1^2) + \frac{648\pi\rho}{5} \\ &= \frac{1548\pi\rho}{5}\,.\end{aligned}$$

THANK YOU