Analysis III: Auditorium Exercise-04

For Engineering Students

Md Tanvir Hassan
University of Hamburg

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The goal is to find the extremal values of a C^1 function

$$f : D \subset \mathbb{R}^n \to \mathbb{R}$$

on the following subset of the domain:

$$G := \{\mathbf{x} \in D \,|\boldsymbol{g}(\mathbf{x}) = \mathbf{0}\} \ \subset \ D,$$

with a C^1 function

$$\boldsymbol{g} : D \to \mathbb{R}^m$$

and $m < n$, i.e., the extremal values must additionally satisfy the m equations

$$\boldsymbol{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))^T = \mathbf{0}$$

Theorem: (Lagrange Multiplier Rule)

Let $\mathbf{x}^0 \in D$ be a local extremum of the function f under the constraint $\boldsymbol{g}(\mathbf{x}^0) = \mathbf{0}$, satisfying the **regularity condition**

$$\text{Rank } \mathbf{J}\boldsymbol{g}(\mathbf{x}^0) = m$$

Then there exist **Lagrange multipliers** $\lambda_1, \ldots, \lambda_m$,
such that the **Lagrange function**

$$F(\mathbf{x}) := f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

satisfies the **necessary first-order condition**:

$$\text{grad } F(\mathbf{x}^0) = \text{grad } f(\mathbf{x}^0) + \sum_{i=1}^{m} \lambda_i \text{grad } g_i(\mathbf{x}^0) = \mathbf{0}.$$

A) Compact Admissible Set and Continuous f
If the admissible set is compact and f is continuous then **Min/Max are attained.**
Candidate with the highest function value = **global maximum**.
Candidate with the smallest function value = **global minimum**.
B) Second-Order Conditions (in the case of two constraints)
Let x_0 be admissible (i.e., $g(x_0) = h(x_0) = 0$), the regularity condition is satisfied at x_0, and assume:

$$\exists \lambda, \mu \text{ with } \nabla F(x_0) = 0.$$

Define the Tangent Space:

$$T_G(x_0) = \{w : \langle w, \nabla g(x_0) \rangle = 0 \text{ and } \langle w, \nabla h(x_0) \rangle = 0\}.$$

Classification: (Min, Max, or Saddle Point)

Necessary for a Local Minimum:

$$w^T H_F(x_0) w \geq 0, \quad \forall w \in T_G(x_0) \setminus \{0\}.$$

Sufficient for a Local Minimum:

$$w^T H_F(x_0) w > 0, \quad \forall w \in T_G(x_0) \setminus \{0\}.$$

Necessary for a Local Maximum:

$$w^T H_F(x_0) w \leq 0, \quad \forall w \in T_G(x_0) \setminus \{0\}.$$

Sufficient for a Local Maximum:

$$w^T H_F(x_0) w < 0, \quad \forall w \in T_G(x_0) \setminus \{0\}.$$

Classification: (Min, Max, or Saddle Point)

This means, in particular, that the necessary conditions for minima (maxima) in the unconstrained case (i.e., positive (negative) semi-definite Hessian matrix) are no longer strictly necessary here.
For example:
The Hessian matrix $H_F(x_0)$ can have negative eigenvalues at a minimum, as long as the corresponding eigenvectors do not represent admissible directions (i.e., directions that lead out of the admissible set).

Compute the extremal values of the function

$$f : \mathbb{R}^2 \to \mathbb{R}\,, \quad f(x, y) = x + y$$

on the circle $x^2 + y^2 = 1$.
a) Under the constraint

$$g(x, y) := x^2 + y^2 - 1 = 0$$

determine the extremal points of the function

$$f(x, y) = x + y$$

using the Lagrange multiplier rule.

Regularity condition:

grad $g(x, y) = (2x, 2y) = (0, 0) \Rightarrow (x, y) = (0, 0)$,

i.e., only $(0, 0)$ violates the regularity condition.

Since $g(0, 0) = -1$, $(0, 0)$ is not on the circle.

All feasible points, i.e., those with $g(x, y) = 0$, satisfy the regularity condition

$$\text{Rank}(\mathbf{J}g(x, y)) = 1.$$

Example: 01 (a)

Lagrangian: $F(x, y) = x + y + \lambda(x^2 + y^2 - 1)$

Lagrange Multiplier Rule:

$$\begin{pmatrix} \nabla F(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda x \\ 1 + 2\lambda y \\ x^2 + y^2 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multiplying the first equation by y and the second by x and subtracting both, we get

$$x - y = 0 \quad \Rightarrow \quad x = y.$$

From the third equation, we then obtain $x^2 + x^2 = 1$

$$\Rightarrow \quad x_{1,2} = \pm\frac{1}{\sqrt{2}}\,, \quad y_{1,2} = \pm\frac{1}{\sqrt{2}}.$$

Extremal candidates:

$$P_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad P_2 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example: 01.(a)

Since the set $g(x, y) = 0$ describes a circle, it is compact.

Thus, the continuous function f attains a maximum and minimum on $g(x, y) = 0$.

We have $f(P_1) = \sqrt{2}$ and $f(P_2) = -\sqrt{2}$.

So, P_1 is a maximum and P_2 is a minimum.

Example: 01.(a)

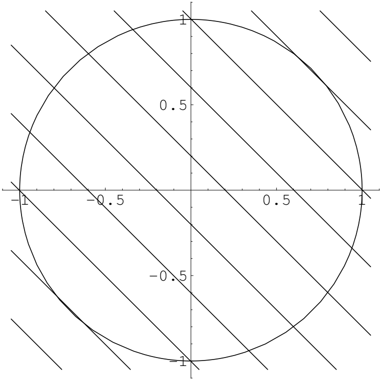

Image: Constraint $g(x, y) = x^2 + y^2 - 1 = 0$
with level curves of the function $f(x, y) = x + y$

Example: 01.(b)

b) Parametrization of the circle

$$g(x, y) := x^2 + y^2 - 1 = 0$$

by c and then solving the extremal problem
for $h(t) := f(c(t))$.

Example: 01.(b)

The circle is parametrized by polar coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} =: c(t)\,, \quad 0 \le t < 2\pi,$$

i.e., $g(\cos t, \sin t) = 0$.

Now, we just need to find the extrema of the function

$$h(t) := f(c(t)) = \cos t + \sin t$$

$$h'(t) = -\sin t + \cos t = 0 \quad \Rightarrow \quad \tan t = 1$$

$$\Rightarrow \quad t_1 = \frac{\pi}{4}\,, \quad t_2 = \frac{5\pi}{4}$$

$$h''(t) = -\cos t - \sin t$$

$$\Rightarrow \quad h''(t_1) = -\sqrt{2} < 0\,,\ h''(t_2) = \sqrt{2}$$

Thus,
$t_1 = \pi/4$ is a maximum with $h(t_1) = \sqrt{2}$
and
$t_2 = 5\pi/4$ is a minimum with $h(t_2) = -\sqrt{2}$.

Example: 01.(b)

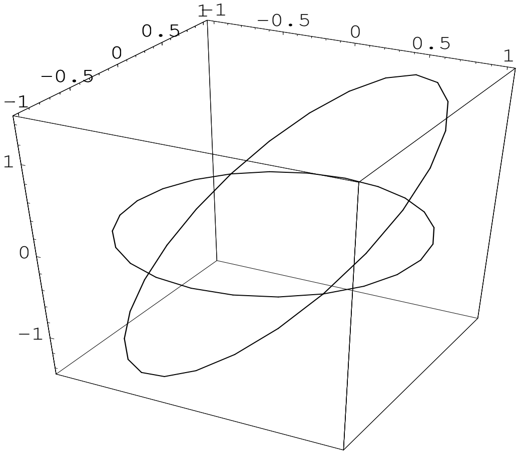

Figure: $c(t)$ and $f(c(t)) = \cos t + \sin t$

For the function

$$f(x, y, z) = z^2$$

compute and classify the extrema on the intersection of the cylinder $x^2 + y^2 = 9$ with the plane $y = z$ using the Lagrange multipliers rule.

Constraints:
$g_1(x, y, z) := x^2 + y^2 - 9 \quad \text{and} \quad g_2(x, y, z) := y - z$.

Regularization condition:

$$Jg(x, y, z) = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

has rank < 2, when the first row is equal to the zero vector, i.e., for the points $(0, 0, z)$.

However, these are not feasible due to

$$g_1(0, 0, z) = -9$$

So, all feasible points satisfy the regularization condition,
The Lagrange multiplier rule can be applied:

<u>Lagrange function</u>:

$$F(x, y, z) = z^2 + \lambda_1(x^2 + y^2 - 9) + \lambda_2(y - z)$$

Lagrange multiplier rule:

$$\begin{pmatrix} \nabla F(x,y,z) \\ g(x,y,z) \end{pmatrix} = \begin{pmatrix} 2\lambda_1 x \\ 2\lambda_1 y + \lambda_2 \\ 2z - \lambda_2 \\ x^2 + y^2 - 9 \\ y - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

1. Equation:

1. Case: $\quad x = 0$

$\Rightarrow \quad 0 = g_1(0, y, z) = y^2 - 9$

$\Rightarrow \quad y = 3 = z \quad \vee \quad y = -3 = z$

Example: 02

Extreme candidates: $P_1 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$

2. Case: $\lambda_1 = 0$

$\Rightarrow \quad \lambda_2 = 0 \quad \Rightarrow \quad z = 0 = y \quad \Rightarrow \quad x = 3 \vee x = -3$

Extreme candidates: $P_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, P_4 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$

The intersection of the cylinder $x^2 + y^2 = 9$ with the plane $y = z$ is an ellipse and therefore compact.

The continuous function f attains its absolute maximum and minimum there.

Among the extreme candidates
are the absolute maximum and minimum.

The function values of the extreme candidates are

$$f(P_{1,2}) = 9\,, \quad f(P_{3,4}) = 0\,.$$

So, $P_{1,2}$ are absolute maxima, and $P_{3,4}$ are absolute minima.

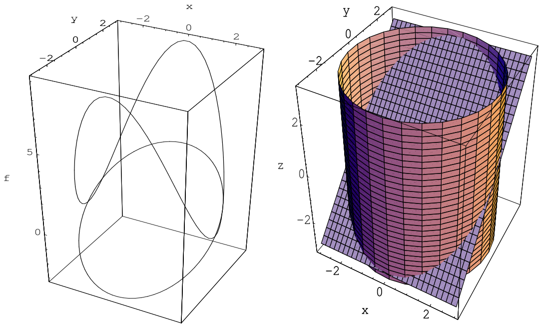

Figure: f on the intersection of the cylinder $x^2 + y^2 = 9$ with the plane $y = z$

Example: 03 [Old Exam Problem]

Determine the global extrema of the function:

$$f(x, y, z) = x - 8y + z$$

on the intersection of the two spherical surfaces:

$$g(x, y, z) = x^2 + (y + 4)^2 + z^2 - 25 = 0$$

and

$$h(x, y, z) = x^2 + y^2 + z^2 - 9 = 0.$$

Regularity Condition (RC):

$$J(g,h)(x,y,z) = \begin{pmatrix} g_x & g_y & g_z \\ h_x & h_y & h_z \end{pmatrix} = \begin{pmatrix} 2x & 2(y+4) & 2z \\ 2x & 2y & 2z \end{pmatrix}$$

RC violated if:

$$\alpha \begin{pmatrix} 2x \\ 2(y+4) \\ 2z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \Rightarrow \begin{cases} \alpha = 1 \vee x = 0 \\ \text{not satisfiable for } \alpha = 1 \\ \alpha = 1 \vee z = 0 \end{cases}$$

Thus, RC can only be violated for $x = z = 0$.

$$g(0,y,0) = 0 + (y+4)^2 + 0 - 25 = 0 \Rightarrow y = -4 \pm 5,$$

$$h(0,y,0) = 0 + y^2 + 0 - 9 = 0 \Rightarrow y = \pm 3.$$

Conclusion: The regularity condition is satisfied at all admissible points.
With $f(x, y, z) = x - 8y + z$ and the Lagrange function:

$$F = f + \lambda g + \mu h,$$

we obtain the necessary conditions for extrema.
1) $F_x = 0$:
2) $F_y = 0$:
3) $F_z = 0$:
4) $g = 0$:

$$x^2 + (y + 4)^2 + z^2 - 25 = 0,$$

5) $h = 0$:

$$x^2 + y^2 + z^2 - 9 = 0.$$

From the last two equations, it follows:

$$(y+4)^2 - y^2 = 16 \iff 8y + 16 = 16 \iff y = 0.$$

Substituting $y = 0$ into the second equation yields $\lambda = 1$, and therefore:

$$\text{I) } 1 + 2x + 2\mu x = 0,$$

$$\text{II) } 1 + 2z + 2\mu z = 0,$$

$$\text{III) } x^2 + z^2 - 9 = 0,$$

$$\lambda = 1,\ y = 0.$$

I - II:

$$(1+\mu)(x-z)=0 \iff \mu=-1 \text{ or } x=z.$$

For $\mu=-1$: (I)
For $x=z$: (III)
Candidates and Function Values For $f(x,y,z)=x-8y+z$:

$$P_1=\begin{pmatrix}\frac{3}{\sqrt{2}}\\ 0\\ \frac{3}{\sqrt{2}}\end{pmatrix}, \quad f(P_1)=3\sqrt{2},$$

$$P_2=\begin{pmatrix}-\frac{3}{\sqrt{2}}\\ 0\\ -\frac{3}{\sqrt{2}}\end{pmatrix}, \quad f(P_2)=-3\sqrt{2}.$$

Since the intersection of the two spherical surfaces (empty, a point, or a circular boundary) is a compact set, the minimum and maximum of the continuous function f are achieved. A comparison of the function values shows that:

- P_1: Global maximum.
- P_2: Global minimum.

Given the optimization problem:

$$f(x,y) = x^2 + y^2 \quad \text{to minimize,}$$

subject to the constraint:

$$g(x,y) = e^{x-1} - \arctan(y+1) - 1 = 0.$$

(a) Verify the regularity condition and demonstrate $x_0 = (1,-1)^T$, along with an appropriate multiplier λ, satisfies the requirements to be an admissible and stationary point of the Lagrange function F.
(b) Investigate the Type of the Stationary Point
Investigate the stationary point $(1,-1)^T$ for its type. For this purpose, set up the Hessian matrix $H_x F(x_0)$ and check its definiteness on the tangent space:

$$\ker Dg(x_0) = T_{G_g}(x_0).$$

Regularity Condition: The Jacobian of $g(x, y)$ is the gradient:

$$Jg(x, y) = \nabla g(x, y) = \left(e^{x-1}, -\frac{1}{1 + (1 + y)^2} \right).$$

For all $(x, y) \in \mathbb{R}^2$, the gradient:

$$\nabla g(x, y) \neq (0, 0).$$

This means the rank of $\nabla g(x, y)$ is 1, satisfying the Regularity Condition.

Admissibility:

The constraint is satisfied at the point $x_0 = (1, -1)^T$:

$$g(1, -1) = e^{1-1} - \arctan(-1 + 1) - 1 = 1 - 0 - 1 = 0.$$

Lagrange Function:

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

where:

$$F(x, y, \lambda) = x^2 + y^2 + \lambda \left(e^{x-1} - \arctan(y+1) - 1\right).$$

Stationary Point:
To find a stationary point of $F(x, y, \lambda)$, we compute:

$$\nabla F(1, -1, \lambda) = 0.$$

Compute the gradient:

$$\nabla F(x, y) = \left(2x + \lambda e^{x-1},\ 2y - \lambda \frac{1}{1 + (1+y)^2}\right).$$

At $x_0 = (1, -1)$:

$$\nabla F(1,-1) = \left(2(1) + \lambda e^{1-1},\ 2(-1) - \lambda \frac{1}{1+(1-1)^2}\right) = (2+\lambda,\ -2-\lambda)\,.$$

Setting $\nabla F(1,-1) = 0$, we find:

$$2 + \lambda = 0 \quad \Rightarrow \quad \lambda = -2.$$

Thus, $x_0 = (1, -1)$ is a stationary point.
Hessian Matrix: The Hessian matrix of $F(x, y)$ is:

$$H_{x,y}F = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix}.$$

Compute the second derivatives:

$$F_{xx} = \frac{\partial}{\partial x}(2x + \lambda e^{x-1}) = 2 + \lambda e^{x-1},$$

$$F_{xy} = F_{yx} = 0,$$

$$F_{yy} = \frac{\partial}{\partial y}\left(2y - \lambda \frac{1}{1+(1+y)^2}\right) = 2 + \lambda \cdot \frac{2(1+y)}{\left(1+(1+y)^2\right)^2}.$$

At $(1, -1)$:

$$F_{xx}(1, -1) = 2 + \lambda e^{1-1} = 2 + \lambda = 0,$$

$$F_{yy}(1, -1) = 2 + \lambda \cdot \frac{2(1-1)}{\left(1+(1-1)^2\right)^2} = 2.$$

Thus:

$$H_{x,y}F(1, -1) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Definiteness on the Tangent Space:
The tangent space is:

$$\ker Dg(x_0) = T_G(x_0) = \left\{ w \in \mathbb{R}^2 : \nabla g(x_0) \cdot w = 0 \right\}.$$

Let $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. The condition:

$$\nabla g(1, -1) \cdot w = e^{1-1} w_1 - \frac{1}{1 + (1-1)^2} w_2 = w_1 - w_2 = 0,$$

implies:

$$w_1 = w_2.$$

On the tangent space, let:

$$w = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \alpha \neq 0.$$

Quadratic Form:
Compute:

$$w^T H_{x,y}F(1,-1)w = \alpha \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Simplify:

$$w^T H_{x,y}F(1,-1)w = \alpha^2 \cdot \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \alpha^2 \cdot 2 = 2\alpha^2 > 0.$$

Since $w^T H_{x,y}F(1,-1)w > 0$ for all $w \neq 0$ in the tangent space, $H_{x,y}F(1,-1)$ is positive definite on the tangent space.
The Hessian is positive definite on the tangent space, confirming that $(1,-1)$ is a strict local minimum of $f(x,y)$ under the constraint $g(x,y) = 0$.

Goal: Find the solution of the system:

$$f(x) = 0, \quad x \in D \subset \mathbb{R}^n, \quad f : D \to \mathbb{R}^n$$

Linearization:
Instead, solve the linear approximation at the point $x^{[k]}$:

$$T_1(x; x^{[k]}) = f(x^{[k]}) + J_f(x^{[k]})(x - x^{[k]}) = 0$$

$$\Rightarrow J_f(x^{[k]})(x - x^{[k]}) = -f(x^{[k]})$$

Use this x as the new approximation and iterate. The initial guess $x^{[0]}$ must be provided.

Iteration:

1. Compute $f(x^{[k]})$.
2. Compute the Jacobian matrix $J_f(x^{[k]})$.
3. Solve the linear system:

$$J_f(x^{[k]}) \cdot \Delta^{[k]} = -f(x^{[k]})$$

4. Update the solution:

$$x^{[k+1]} = x^{[k]} + \Delta^{[k]}$$

Initial Guess: Start with $x^{[0]}$ and iterate until convergence.

Example: 05

To determine the minimum of the function:

$$z = F(x, y) := x^2 + 2y^2 - 0.1\cos(x + y) - 3x + 2y,$$

Newton's Method should be applied to the function $f(x, y) := \nabla F(x, y)^T$.

(a) Compute $f(x, y)$ and the Jacobian matrix $J_f(x, y)$.

(b) Formulate Newton's Method and perform the first iteration step by hand, using the starting vector:

$$(x_0, y_0) = (0, 0).$$

(c) Perform the iteration numerically and compute the solution to at least ten decimal places of accuracy.

$$f(x,y) = \begin{pmatrix} 2x + 0.1\sin(x+y) - 3 \\ 4y + 0.1\sin(x+y) + 2 \end{pmatrix}$$

$$f(0,0) = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$J_f(x,y) = \begin{pmatrix} 2 + 0.1 * \cos(x+y) & 0.1 * \cos(x+y) \\ 0.1 * \cos(x+y) & 4 + 0.1 * \cos(x+y) \end{pmatrix}$$

$$J_f(0,0) = \begin{pmatrix} 2.1 & 0.1 \\ 0.1 & 4.1 \end{pmatrix}$$

$$J_f(0,0) \cdot \Delta^{[0]} = -f(0,0) \iff \begin{pmatrix} 2.1 & 0.1 \\ 0.1 & 4.1 \end{pmatrix} \cdot \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} = - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$x^{[1]} = x^{[0]} + \Delta^{[0]}$$

```
format long
k = 0;
x = 0; % Initial value
y = 0; % Initial value
for j = 1:5 % Number of Newton iterations
    c = 0.1 * cos(x + y);
    s = 0.1 * sin(x + y);
    A = [2 + c c; c 4 + c]; % Jacobian matrix
    b = [-2 * x - s + 3; -4 * y - s - 2]; % Function whose root is sought
    % Newton iteration
    dx = A \ b; % Solve the linear system
    k = k + 1;
    x = x + dx(1, 1);
    y = y + dx(2, 1);
end
% Gradient of the original function
g = b;
```

Results of Iterations

Iteration (k)	x	y
0	0.00000000000000	0.00000000000000
1	1.45348837209302	−0.52325581395349
2	1.45963647269063	−0.52018176365468
3	1.45963810885761	−0.52018094557119
4	1.45963810885773	−0.52018094557114
5	1.45963810885773	−0.52018094557114

Gradient of F at point $x^{[5]}$:

$$g = 10^{-15} \cdot (-0.444089209850063, -0.222044604925031)$$

Since the Hessian matrix of F (i.e., the Jacobian matrix of f) is positive definite, this is indeed an approximation of a minimum.

THANK YOU