Analysis III: Auditorium Exercise-03

For Engineering Students

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Let's consider a function f,

$$\begin{aligned} f : D \subset \mathbb{R}^n &\rightarrow \mathbb{R} \\ x &\mapsto f(x) \end{aligned}$$

where $x = (x_1, \cdots, x_n)$.

Definition: For $x^0 \in D$, we define

- f has a **global maximum** at x^0 if for all $x \in D$, $f(x) \leq f(x^0)$.
- f has a **local maximum** at x^0 if there exists $\varepsilon > 0$ such that for all $x \in D$ with $||x - x^0|| < \varepsilon$, $f(x) \leq f(x^0)$.

- If the inequality $f(x) \le f(x^0)$ can be replaced by $f(x) < f(x^0)$ for $x \ne x^0$, it is a **strict maximum** at x^0.
- If $f(x) \ge f(x^0)$, and $f(x) > f(x^0)$, then it is a **minimum** at x^0.
- f has an **extremum** at x^0 if it is either a maximum or minimum.
- f has a **stationary point** at $x^0 \in D$ if grad $f(x^0) = 0$.

Theorem: First-Order Necessary Condition

Let f be a C^1 function in D^0, and $x^0 \in D^0$ is a **local extremum**, then

$$\operatorname{grad} f(x^0) = 0.$$

For a twice-partially differentiable function,

$$Hf(x) = \begin{pmatrix} f_{x_1x_1}(x) & \cdots & f_{x_1x_n}(x) \\ \vdots & & \vdots \\ f_{x_nx_1}(x) & \cdots & f_{x_nx_n}(x) \end{pmatrix}$$

represents the **Hessian matrix** of f.

Theorem: Second-Order Necessary Condition

If f is a C^2 function and $x^0 \in D^0$ is a stationary point, then:

1. If $x^0 \in D$ is a **local minimum**,
   then $Hf(x^0)$ is positive semidefinite.

2. If $x^0 \in D$ is a **local maximum**,
   then $Hf(x^0)$ is negative semidefinite.

Theorem: Second-Order Sufficient Condition

If f is a C^2 function and $x^0 \in D^0$ is a stationary point, then:

1. If $Hf(x^0)$ is positive definite,
   then x^0 is a strict local minimum.

2. If $Hf(x^0)$ is negative definite,
   then x^0 is a strict local maximum.

3. If $Hf(x^0)$ is indefinite,
   then x^0 is a **saddle point**.

Example: 01

Compute all stationary points of the following function and classify them:

$$f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$$

grad $f(x,y) = e^{-x^2-y^2}(2x(1-x^2+y^2), 2y(-1-x^2+y^2))^T = (0,0)^T$
To compute the stationary points, we set $f_x(x,y) = 0$ and consider all cases.
Case 1: $x = 0$
$\Rightarrow \; 0 = f_y(0,y) = e^{-y^2} 2y(-1+y^2)$
$\Rightarrow \quad y = 0, \quad y = 1, \quad y = -1$
$\Rightarrow$ stationary points:

$$P_1 = (0,0), \quad P_2 = (0,1), \quad P_3 = (0,-1)$$

Case 2: $1 - x^2 + y^2 = 0 \Rightarrow \; x^2 = 1 + y^2$
$\Rightarrow \; 0 = f_y(x,y) = e^{-(1+y^2)-y^2} 2y(-1-(1+y^2)+y^2)$
$\quad = -4ye^{-1-2y^2}$
$\Rightarrow \; y = 0 \quad \Rightarrow \quad x = 1, \quad x = -1$
$\Rightarrow$ stationary points: $P_4 = (1,0), \quad P_5 = (-1,0)$

$$Hf(x,y) = $$
$$2e^{-x^2-y^2}\begin{pmatrix} 1-5x^2+2x^4+y^2-2x^2y^2 & 2xy(x^2-y^2) \\ 2xy(x^2-y^2) & -1+5y^2-2y^4-x^2+2x^2y^2 \end{pmatrix}$$

$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ is indefinite

$\Rightarrow \quad P_1 = (0,0)$ is a saddle point.

$Hf(0,\pm 1) = 2e^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is positive definite

$\Rightarrow \; P_{2,3} = (0,\pm 1)$ are minima.

$Hf(\pm 1,0) = -2e^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is negative definite

$\Rightarrow \; P_{4,5} = (\pm 1,0)$ are maxima.

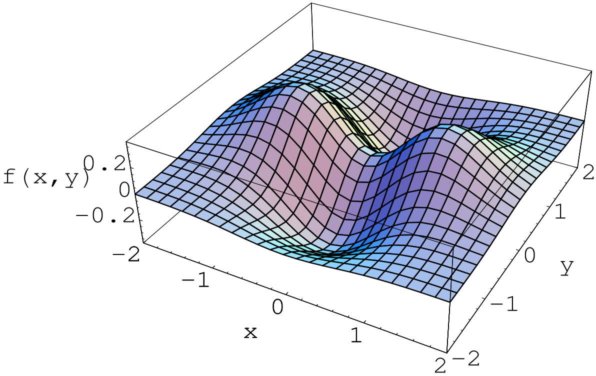


Figure : $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$

Example: 02

Compute all stationary points of the following function and classify them:

$$f(x, y) = y(y^2 - 3)$$

grad $f(x,y) = (0, 3y^2 - 3)^T = (0,0)^T \;\Rightarrow\; y = \pm 1,\ x \in \mathbb{R}$
The stationary points lie on the lines
$P_1(x) = (x, 1)$ and $P_2(x) = (x, -1)$.

$$Hf(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 6y \end{pmatrix}$$

$Hf(x,1) = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$ is positive semi definite
$\Rightarrow \quad P_1(x) = (x,1)$ are not local maxima.
$Hf(x,-1) = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}$ is negative semidefinite
$\Rightarrow \quad P_2(x) = (x,-1)$ are not local minima.
f is independent of x,
i.e., for fixed $y = c$, $f(x,c) = \mathit{constant}$ for all $x \in \mathbb{R}$.

The extrema are thus the ones of $g(y) = y(y^2 - 3)$,
i.e., all points on the line $P_1(x) = (x, 1)$ are local minima and for $P_2(x) = (x, -1)$ one obtains local maxima.

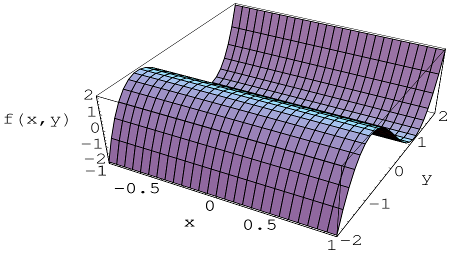


Figure : $f(x, y) = y(y^2 - 3)$

Compute all stationary points of the following function and classify them:

$$f(x, y) = \sin(x^2 + y^2)$$

Solution: grad $f(x, y) = 2\cos(x^2 + y^2)(x, y)^T = (0, 0)^T$
The stationary points are thus given by $(0, 0)$ and all points P, for which $x^2 + y^2 = \pi/2 + n\pi$ with $n \in \mathbb{N}_0$.
$Hf(x, y) =$
$$\begin{pmatrix} 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2) & -4xy\sin(x^2+y^2) \\ -4xy\sin(x^2+y^2) & 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2) \end{pmatrix}$$
$Hf(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is positive definite
$\Rightarrow \quad (0, 0)$ is a minimum.
$Hf(P) = \begin{pmatrix} -4x^2\sin(x^2+y^2) & -4xy\sin(x^2+y^2) \\ -4xy\sin(x^2+y^2) & -4y^2\sin(x^2+y^2) \end{pmatrix}$
is semi definite, as $\det Hf(P) = 0$.

We classify differently:
For points P on the circles $x^2 + y^2 = \pi/2 + n\pi$
we have $\sin(\pi/2 + n\pi) = (-1)^n$.
Therefore, for even n there are maxima, and for odd n there are minima on these circles.

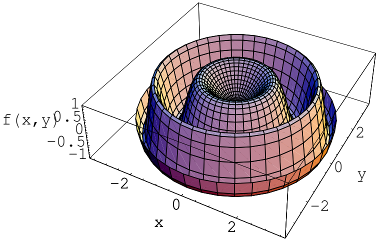


Figure : $f(x, y) = \sin(x^2 + y^2)$

Given the function

$$f(x, y) = 8x^4 - 10x^2y + 3y^2.$$

1. Calculate all stationary points of f
2. Try to apply the sufficient condition for the classification of stationary points.
3. Show that f has a local minimum at the origin along every line through the origin.
4. Does f also have a minimum at the origin along every parabola $y = ax^2$ with $a \in \mathbb{R}$?
5. Plot the function, for example, using the MATLAB routines 'ezsurf' and 'ezcontour'.

Solution:

grad $f(x,y) = (4x(8x^2 - 5y), -10x^2 + 6y)^T = 0$

1. Case: $\quad x = 0$

$\Rightarrow \; 6y = 0 \quad \Rightarrow \quad$ stationary point $(x_0, y_0) = (0,0)$.

2. Case: $\quad 8x^2 - 5y = 0$

$\Rightarrow \quad y = 8x^2/5 \;\Rightarrow \quad -10x^2 + 6 \cdot 8x^2/5 = 0 \;\Rightarrow \quad x = 0$ The only stationary point is thus $(0,0)$.

$Hf(x,y) = \begin{pmatrix} 96x^2 - 20y & -20x \\ -20x & 6 \end{pmatrix} \quad \Rightarrow \quad Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$ is positive semi definite,
and the sufficient criterion is not applicable.
The necessary condition of order 2 leaves the possibilities of being a minimum or a saddle point for the stationary point

$$(x_0, y_0) = (0,0).$$

On the line $x = 0$, the function is described by

$$g(y) := f(0, y) = 3y^2.$$

For $y = 0$, g has a strict local minimum. All other origin lines can be represented by $y = ax$ with $a \in \mathbb{R}$ and the function is then described by

$$h(x) := f(x, ax) = 8x^4 - 10ax^3 + 3a^2x^2$$

For $a = 0$, h is minimal at $x = 0$. For $a \neq 0$, a minimum is also obtained at $x = 0$ because

$$h'(x) = 32x^3 - 30ax^2 + 6a^2x \quad \Rightarrow \quad h'(0) = 0$$

and

$$h''(x) = 96x^2 - 60ax + 6a^2 \quad \Rightarrow \quad h''(0) = 6a^2 > 0\,.$$

On the parabola $y = ax^2$, the function takes the form
$p(x) := f(x, ax^2) = 8x^4 - 10ax^4 + 3a^2x^4$
$= x^4(3a^2 - 10a + 8) = x^4(a-2)(3a-4)$. This yields

$$
\begin{array}{rclcrcl}
p'(x) & = & 4x^3(a-2)(3a-4) & \Rightarrow & p'(0) & = & 0 \\
p''(x) & = & 12x^2(a-2)(3a-4) & \Rightarrow & p''(0) & = & 0 \\
p'''(x) & = & 24x(a-2)(3a-4) & \Rightarrow & p'''(0) & = & 0 \\
p''''(x) & = & 24(a-2)(3a-4) & \Rightarrow & p''''(0) & = & 24(a-2)(3a-4)\,.
\end{array}
$$

For $a \in \,]4/3, 2[$, $p''''(0) < 0$
and there is a strict maximum at $x = 0$.
For $a \notin \,[4/3, 2]$, $p''''(0) > 0$
and there is a strict minimum at $x = 0$.

Example: 04.(4)

Thus, at the stationary point $(0, 0)$, it is a saddle point. If it were known that

$$f(x, y) = (2y - 3x^2)^2 - (y - x^2)^2$$

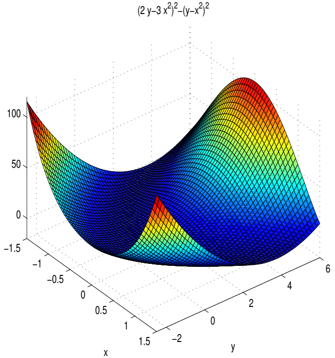
, then on the origin parabola

$$2y - 3x^2 = 0$$

at $x = 0$, an immediate maximum
and on

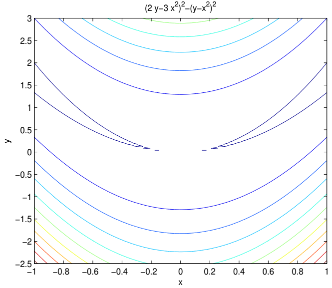
$$y - x^2 = 0$$

at $x = 0$, an immediate minimum would have been recognized and then it would have been immediately inferred to be a saddle point.



ezsurf('8*x$^4 - 10*x^2*y + 3*y^2$', $[-1.5, 1.5, -2.5, 6]$)

Figure: $f(x, y) = 8x^4 - 10x^2y + 3y^2$



$$\text{ezcontour('8*x}^4 - 10 * x^2 * y + 3 * y^{2\prime}, [-1, 1, -2.5, 3])$$

Figure: $f(x, y) = 8x^4 - 10x^2y + 3y^2$

The solvability of the system of equations is examined:

$$\begin{aligned} g_1(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0 \\ &\vdots \\ g_m(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0\,, \end{aligned}$$

briefly denoted as $g(x, y) = 0$, for the variable $y \in \mathbb{R}^m$.

In this case, y would be expressible as a function of x,

In the equation $g(x, y) = 0$, the function f would be implicitly contained.

Let $g : D \to \mathbb{R}^m$ be a C^1 function defined on the open set and the set $D \subset \mathbb{R}^n \times \mathbb{R}^m$, and consider a point $(x^0, y^0) \in D$ where $x^0 \in \mathbb{R}^n$ and $y^0 \in \mathbb{R}^m$ such that $g(x^0, y^0) = 0$.

Furthermore, assume that the following $m \times m$ submatrix of $Jg(x^0, y^0)$ is regular:

$$\frac{\partial g}{\partial y}(x^0, y^0) := \begin{pmatrix} \frac{\partial g_1}{\partial y_1}(x^0, y^0) & \cdots & \frac{\partial g_1}{\partial y_m}(x^0, y^0) \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1}(x^0, y^0) & \cdots & \frac{\partial g_m}{\partial y_m}(x^0, y^0) \end{pmatrix}$$

Then there exist open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ with $x^0 \in U$, $y^0 \in V$, and $U \times V \subset D$, and a uniquely determined continuously differentiable function:

$$f : U \to V$$

such that,

$$y^0 = f(x^0) \quad \text{and} \quad g(x, f(x)) = 0 \quad \text{for all} \quad x \in U \, .$$

The Jacobian matrix Jf is computed for all $x \in U$ by differentiating the implicit equation $g(x, f(x)) = 0$ (using the chain rule), which leads to the equation system:

$$\frac{\partial g}{\partial x}(x, f(x)) + \frac{\partial g}{\partial y}(x, f(x)) \cdot Jf(x) = 0.$$

For a C^1-function $g : \mathbb{R}^2 \to \mathbb{R}$, the solution set given by

$$g(x, y) = 0$$

is examined.

The solvability of the equation for one of the variables is guaranteed when $g_x \neq 0$ or $g_y \neq 0$, that is,

$$\operatorname{grad} g = (g_x, g_y) \neq 0$$

The points (x_0, y_0) for which grad $g(x_0, y_0) \neq 0$ are therefore called **regular**.
In regular points, the solution set

$$g = 0$$

is described by a contour line.
In this context, a **horizontal tangent** is present at (x_0, y_0) if

$$g(x_0, y_0) = 0, \quad g_x(x_0, y_0) = 0, \quad g_y(x_0, y_0) \neq 0$$

holds, and a **vertical tangent** for

$$g(x_0, y_0) = 0, \quad g_x(x_0, y_0) \neq 0, \quad g_y(x_0, y_0) = 0.$$

The points (x_0, y_0) for which grad $g(x_0, y_0) = 0$ are called **singular** or **stationary**.

Classification of singular points of $g(x, y) = 0$:

(x_0, y_0) is an **isolated point** if $\det Hg(x_0, y_0) > 0$,

(x_0, y_0) is a **double point** if $\det Hg(x_0, y_0) < 0$.

(x_0, y_0) is a **cusp point** if $\det Hg(x_0, y_0) = 0$.

To investigate the curve implicitly defined by the level set

$$f(x, y) := x^3 + y^3 - xy = 0,$$

we follow the instructions provided.

a) Determine the symmetries of the curve.

The curve is symmetric with respect to the bisector, meaning that $f(x, y) = f(y, x)$. We recall the reflection matrix S_α:

$$\underbrace{\begin{pmatrix} \cos\left(\frac{2 \cdot \pi}{4}\right) & \sin\left(\frac{2 \cdot \pi}{4}\right) \\ \sin\left(\frac{2 \cdot \pi}{4}\right) & -\cos\left(\frac{2 \cdot \pi}{4}\right) \end{pmatrix}}_{=S_{\pi/4}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

This reflects the point (x, y) across the line $y = x$.

b) Determine the points on the curve with a horizontal tangent.

$\operatorname{grad} f(x, y) = (3x^2 - y, 3y^2 - x)^T$

Points on the curve with a horizontal tangent are obtained from the conditions

$$f_x(x, y) = 0 \quad \wedge \quad f(x, y) = 0 \quad \wedge \quad f_y(x, y) \neq 0$$

$$0 = f_x(x,y) = 3x^2 - y \quad \Rightarrow \quad y = 3x^2 \quad \Rightarrow$$

$$0 = f(x, 3x^2) = x^3 + (3x^2)^3 - x3x^2 = x^3(27x^3 - 2)$$

$$\Rightarrow \quad x = 0 \quad \vee \quad x = \frac{2^{1/3}}{3}$$

$$\Rightarrow \quad P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; P_1 = \frac{1}{3}\begin{pmatrix} 2^{1/3} \\ 2^{2/3} \end{pmatrix}.$$

Only for P_1 does the condition $f_y(P_1) \neq 0$ hold.

Therefore, P_1 is a point with a horizontal tangent.

c) Determine the points on the curve with a vertical tangent.

Points on the curve with a vertical tangent are obtained from the conditions

$$f_y(x,y) = 0 \quad \wedge \quad f(x,y) = 0 \quad \wedge \quad f_x(x,y) \neq 0.$$

$$0 = f_y(x,y) = 3y^2 - x \quad \Rightarrow \quad x = 3y^2 \quad \Rightarrow$$
$$0 = f(3y^2, y) = (3y^2)^3 + y^3 - 3y^2 y = y^3(27y^3 - 2)$$
$$\Rightarrow \quad y = 0 \quad \vee \quad y = \frac{2^{1/3}}{3}$$

$$\Rightarrow \quad P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; P_2 = \frac{1}{3}\begin{pmatrix} 2^{2/3} \\ 2^{1/3} \end{pmatrix}$$

Only for P_2 does the condition $f_x(P_2) \neq 0$ hold.

Therefore, P_2 is a point with a vertical tangent.

This can also be deduced without calculation from the symmetry.

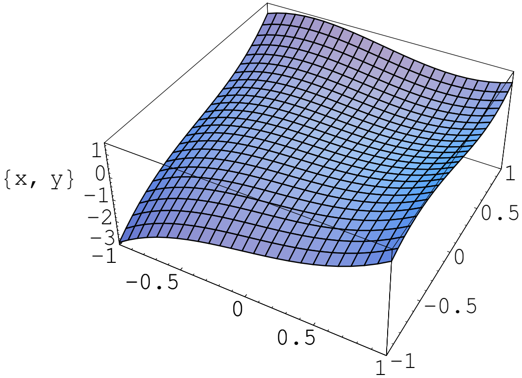
d) Classify the singular points of the curve.

For $P_0 = (0,0)^T$, $\operatorname{grad} f(0,0) = \mathbf{0}$, making P_0 a singular point.

$$Hf(x,y) = \begin{pmatrix} 6x & -1 \\ -1 & 6y \end{pmatrix} \quad \Rightarrow \quad Hf(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Since $\det Hf(0,0) = -1 < 0$, P_0 is a double point.

e) Draw the level set:



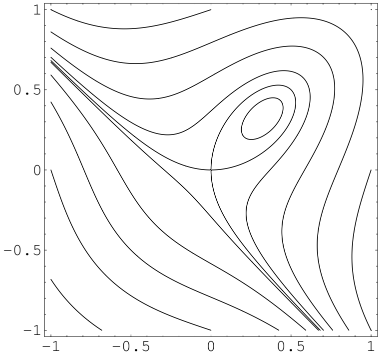


Figure: $f(x, y) = x^3 + y^3 - xy = c$
for $c = -2, -1, -0.5, -0.2, -0.025, 0, 0.05, 0.2, 0.5, 1$

THANK YOU