Analysis III: Auditorium Exercise-02

For Engineering Students

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Consider a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ that is m times continuously partially differentiable in D, where D is open and convex, and $n, m \in \mathbb{N}$. Let $x^0 \in D$. Then the Taylor expansion of f at x^0 up to order m is defined as:

$$T_m(x; x^0) := \sum_{j=0}^{m} \frac{1}{j!} \left(\left((x - x^0)^T \nabla \right)^j f \right) (x^0)$$

Hessian Matrix

The **Hessian matrix** is a matrix that represents the second-order partial derivatives of a multivariable function. For a function $f : \mathbb{R}^n \to \mathbb{R}$, the Hessian matrix at a point $\mathbf{x}$ is denoted by $Hf(\mathbf{x})$ and is defined as:

$$Hf(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}.$$

Each entry H_{ij} in the Hessian matrix represents a second partial derivative of f, where $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

- **Symmetry**: If f is twice continuously differentiable, the Hessian matrix is symmetric, meaning $H_{ij} = H_{ji}$.

$$
\begin{aligned}
& T_2(x, y, z; x_0, y_0, z_0) \\
= \; & f(x_0, y_0, z_0) \\
& + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \\
& + \frac{1}{2}\left(f_{xx}(x_0, y_0, z_0)(x - x_0)^2 + f_{yy}(x_0, y_0, z_0)(y - y_0)^2\right. \\
& + f_{zz}(x_0, y_0, z_0)(z - z_0)^2 + 2f_{xy}(x_0, y_0, z_0)(x - x_0)(y - y_0) \\
& \left. + 2f_{xz}(x_0, y_0, z_0)(x - x_0)(z - z_0) + 2f_{yz}(x_0, y_0, z_0)(y - y_0)(z - z_0)\right)
\end{aligned}
$$

$$
\begin{aligned}
& T_3(x, y; x_0, y_0) \\
= \quad & f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
& + \tfrac{1}{2}\left(f_{xx}(x_0, y_0)(x - x_0)^2 + 2 f_{xy}(x_0, y_0)(x - x_0)(y - y_0)\right. \\
& \qquad \left. + f_{yy}(x_0, y_0)(y - y_0)^2\right) \\
& + \tfrac{1}{6}\left(f_{xxx}(x_0, y_0)(x - x_0)^3 + 3 f_{xxy}(x_0, y_0)(x - x_0)^2 (y - y_0)\right. \\
& \qquad \left. + 3 f_{xyy}(x_0, y_0)(x - x_0)(y - y_0)^2 + f_{yyy}(x_0, y_0)(y - y_0)^3\right)
\end{aligned}
$$

Calculate the Taylor polynomial of degree 2 for the following function

$$f(x, y, z) = 1 + z + xy + x^2(1 - y)^2 + (y + z)^3$$

around the expansion point $(0, 0, 0)$.

Solution:

$$f(x,y,z) = 1 + z + xy + x^2(1-y)^2 + (y+z)^3 \quad \Rightarrow \quad f(0,0,0) = 1$$

$$\begin{aligned}
f_x(x,y,z) &= y + 2x(1-y)^2 &&\Rightarrow \quad f_x(0,0,0) = 0\\
f_y(x,y,z) &= x - 2x^2(1-y) + 3(y+z)^2 &&\Rightarrow \quad f_y(0,0,0) = 0\\
f_z(x,y,z) &= 1 + 3(y+z)^2 &&\Rightarrow \quad f_z(0,0,0) = 1
\end{aligned}$$

$$\begin{aligned}
f_{xx}(x,y,z) &= 2(1-y)^2 &&\Rightarrow \quad f_{xx}(0,0,0) = 2\\
f_{xy}(x,y,z) &= 1 - 4x(1-y) &&\Rightarrow \quad f_{xy}(0,0,0) = 1\\
f_{xz}(x,y,z) &= 0 &&\Rightarrow \quad f_{xz}(0,0,0) = 0
\end{aligned}$$

$$\begin{aligned}
f_{yy}(x,y,z) &= 2x^2 + 6(y+z) &&\Rightarrow \quad f_{yy}(0,0,0) = 0\\
f_{yz}(x,y,z) &= 6(y+z) &&\Rightarrow \quad f_{yz}(0,0,0) = 0\\
f_{zz}(x,y,z) &= 6(y+z) &&\Rightarrow \quad f_{zz}(0,0,0) = 0
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow T_2(x,y,z;0,0,0) &= f(0,0,0) + f_x(0,0,0)x + f_y(0,0,0)y + f_z(0,0,0)z \\
&+\tfrac{1}{2}\left(f_{xx}(0,0,0)x^2 + f_{yy}(0,0,0)y^2 + f_{zz}(0,0,0)z^2\right. \\
&\left.+2f_{xy}(0,0,0)xy + 2f_{xz}(0,0,0)xz + 2f_{yz}(0,0,0)yz\right) \\
&= 1 + z + xy + x^2
\end{aligned}
$$

Since the expansion point is the origin, it would have been easier to expand the given function by multiplication and then omit terms beyond the quadratic ones:

$$f(x,y,z) = 1 + z + xy + x^2 - 2yx^2 + x^2y^2 + y^3 + 3y^2z + 3yz^2 + z^3 \,.$$

Find the 3rd-degree Taylor polynomial of the following function

$$f(x, y) = x\sin(x + y)$$

at the point $\left(0, \frac{\pi}{2}\right)$.

Solution:

$$
\begin{aligned}
f(x,y) &= x\sin(x+y) && \Rightarrow\quad f\left(0,\tfrac{\pi}{2}\right)=0\\
f_x(x,y) &= \sin(x+y)+x\cos(x+y) && \Rightarrow\quad f_x\left(0,\tfrac{\pi}{2}\right)=1\\
f_y(x,y) &= x\cos(x+y) && \Rightarrow\quad f_y\left(0,\tfrac{\pi}{2}\right)=0
\end{aligned}
$$

$$
\begin{aligned}
f_{xx}(x,y) &= 2\cos(x+y)-x\sin(x+y) && \Rightarrow\quad f_{xx}\left(0,\tfrac{\pi}{2}\right)=0\\
f_{xy}(x,y) &= \cos(x+y)-x\sin(x+y) && \Rightarrow\quad f_{xy}\left(0,\tfrac{\pi}{2}\right)=0\\
f_{yy}(x,y) &= -x\sin(x+y) && \Rightarrow\quad f_{yy}\left(0,\tfrac{\pi}{2}\right)=0
\end{aligned}
$$

$$
\begin{aligned}
f_{xxx}(x,y) &= -3\sin(x+y)-x\cos(x+y) && \Rightarrow\quad f_{xxx}(0,\pi/2)=-3\\
f_{xxy}(x,y) &= -2\sin(x+y)-x\cos(x+y) && \Rightarrow\quad f_{xxy}(0,\pi/2)=-2\\
f_{xyy}(x,y) &= -\sin(x+y)-x\cos(x+y) && \Rightarrow\quad f_{xyy}(0,\pi/2)=-1\\
f_{yyy}(x,y) &= -x\cos(x+y) && \Rightarrow\quad f_{yyy}(0,\pi/2)=0
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow T_3(x,y;0,\pi/2) &= f(0,\pi/2) + f_x(0,\pi/2)x + f_y(0,\pi/2)(y-\pi/2) \\
&\quad + \frac{1}{2}\left(f_{xx}(0,\pi/2)x^2 + 2f_{xy}(0,\pi/2)x(y-\pi/2)\right. \\
&\qquad \left. + f_{yy}(0,\pi/2)(y-\pi/2)^2\right) \\
&\quad + \frac{1}{6}\left(f_{xxx}(0,\pi/2)x^3 + 3f_{xxy}(0,\pi/2)x^2(y-\pi/2)\right. \\
&\qquad + 3f_{xyy}(0,\pi/2)x(y-\pi/2)^2 + f_{yyy}(0,\pi/2)(y-\pi/2 \\
&= x - x^3/2 - x^2(y-\pi/2) - x(y-\pi/2)^2/2
\end{aligned}$$

If f is $(m+1)$ times continuously partially differentiable, then for the **Taylor expansion**

$$f(x) = T_m(x; x^0) + R_m(x; x^0)$$

the following **Lagrange remainder formula** holds, with $\xi := x^0 + \Theta(x - x^0)$ and $0 < \Theta < 1$

$$R_m(x; x_0) = \frac{1}{(m+1)!} \left(\left((x - x_0)^T \nabla \right)^{(m+1)} f \right)(\xi)$$

Alternatively, in terms of **multi-indices**:

$$R_m(x; x^0) = \sum_{|\alpha|=m+1} \frac{D^\alpha f(\xi)}{\alpha!} (x - x^0)^\alpha \, .$$

$$
\begin{aligned}
R_3(x,y;x_0,y_0) \quad = \quad & +\tfrac{1}{4!}\Big(f_{xxxx}(\xi_1,\xi_2)(x-x_0)^4 \\
& +4f_{xxxy}(\xi_1,\xi_2)(x-x_0)^3(y-y_0) \\
& +6f_{xxyy}(\xi_1,\xi_2)(x-x_0)^2(y-y_0)^2 \\
& +4f_{xyyy}(\xi_1,\xi_2)(x-x_0)(y-y_0)^3 \\
& +f_{yyyy}(\xi_1,\xi_2)(y-y_0)^4\Big)
\end{aligned}
$$

Calculate the 2nd-degree Taylor Polynomial for the point of development $(x_0, y_0) = (0, 0)$ for the following function

$$h(x, y) = \cos(x^2 + y^2)$$

and estimate the error that arises when using T_2 instead of h in the rectangle $[0, \pi/4] \times [0, \pi/4]$, from above.

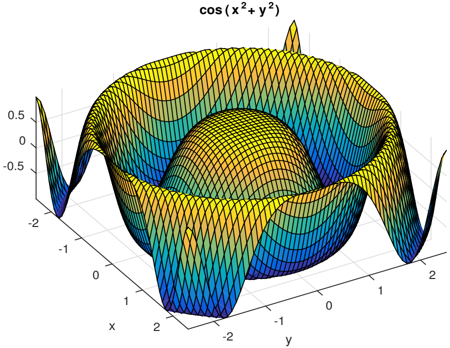
Solution:

$$\begin{aligned}
&h(x,y) = \cos(x^2+y^2) &&\Rightarrow\quad h(0,0)=1\\
&h_x(x,y) = -2x\sin(x^2+y^2) &&\Rightarrow\quad h_x(0,0)=0\\
&h_y(x,y) = -2y\sin(x^2+y^2) &&\Rightarrow\quad h_y(0,0)=0\\
&h_{xx}(x,y) = -2\sin(x^2+y^2) - 4x^2\cos(x^2+y^2) &&\Rightarrow\quad h_{xx}(0,0)=0\\
&h_{xy}(x,y) = -4xy\cos(x^2+y^2) &&\Rightarrow\quad h_{xy}(0,0)=0\\
&h_{yy}(x,y) = -2\sin(x^2+y^2) - 4y^2\cos(x^2+y^2) &&\Rightarrow\quad h_{yy}(0,0)=0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad T_2(x,y;0,0) &= h(0,0) + h_x(0,0)x + h_y(0,0)y\\
&\quad + \tfrac{1}{2}\left(h_{xx}(0,0)x^2 + h_{xy}(0,0)xy + h_{yy}(0,0)y^2\right)\\
&= 1
\end{aligned}$$

MATLAB command for surface plot:

ezsurf('cos($x^2 + y^2$)', [−2.5, 2.5, −2.5, 2.5])



For the error estimation, the third derivatives are required:

$$\begin{aligned}
h_{xxx}(x,y) &= -12x\cos(x^2+y^2)+8x^3\sin(x^2+y^2)\\
h_{xxy}(x,y) &= -4y\cos(x^2+y^2)+8x^2y\sin(x^2+y^2)\\
h_{xyy}(x,y) &= -4x\cos(x^2+y^2)+8y^2x\sin(x^2+y^2)\\
h_{yyy}(x,y) &= -12y\cos(x^2+y^2)+8y^3\sin(x^2+y^2).
\end{aligned}$$

The error estimation for any $(x,y)\in[0,\pi/4]\times[0,\pi/4]$ implies, with $\theta\in]0,1[$, any

$$(\xi_1,\xi_2):=(0,0)+\theta(x,y)\in]0,\pi/4[\times]0,\pi/4[$$

Using the triangle inequality, we obtain:

$$\begin{aligned} & |h(x,y) - T_2(x,y;0,0)| = |R_2(x,y;0,0)| \\ = \; & \tfrac{1}{3!} \left| h_{xxx}(\xi_1,\xi_2)x^3 + 3h_{xxy}(\xi_1,\xi_2)x^2y + 3h_{xyy}(\xi_1,\xi_2)xy^2 + h_{yyy}(\xi_1,\xi_2)y^3 \right| \\ \leq \; & \tfrac{1}{3!} \big(|h_{xxx}(\xi_1,\xi_2)| \cdot |x|^3 + 3\,|h_{xxy}(\xi_1,\xi_2)| \cdot |x^2y| \\ & +3\,|h_{xyy}(\xi_1,\xi_2)| \cdot |xy^2| + |h_{yyy}(\xi_1,\xi_2)| \cdot |y^3| \big) . \end{aligned}$$

Each of the four terms can now be individually upper-bounded. Using $|\sin t| \leq 1$ and $|\cos t| \leq 1$, we have:

$$\begin{aligned} & |h_{xxx}(\xi_1,\xi_2)| \cdot |x|^3 \\ = \; & \left| -12\xi_1 \cos(\xi_1^2+\xi_2^2) + 8\xi_1^3 \sin(\xi_1^2+\xi_2^2) \right| \cdot |x|^3 \\ \leq \; & \big(|-12\xi_1| \cdot |\cos(\xi_1^2+\xi_2^2)| + |8\xi_1^3| \cdot |\sin(\xi_1^2+\xi_2^2)| \big) \cdot |x|^3 \\ \leq \; & \left(12 \cdot \frac{\pi}{4} + 8 \cdot \left(\frac{\pi}{4}\right)^3 \right) \left(\frac{\pi}{4}\right)^3 \end{aligned}$$

Similarly,

$$3\,|h_{xxy}(\xi_1,\xi_2)|\cdot|x^2y| \le 3\left(4\cdot\frac{\pi}{4}+8\cdot\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$

$$3\,|h_{xyy}(\xi_1,\xi_2)|\cdot|xy^2| \le 3\left(4\cdot\frac{\pi}{4}+8\cdot\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$

$$|h_{yyy}(\xi_1,\xi_2)|\cdot|y^3| \le \left(12\cdot\frac{\pi}{4}+8\cdot\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$ Overall, we have:

$$|h(x,y)-T_2(x,y;0,0)| \le \frac{\pi^3}{3!4^3}\left(48\cdot\frac{\pi}{4}+64\cdot\left(\frac{\pi}{4}\right)^3\right) = 5.5476...$$

The maximum error occurs at $x = y = \dfrac{\pi}{4}$:

$$|h\left(\frac{\pi}{4},\frac{\pi}{4}\right)-T_2\left(\frac{\pi}{4},\frac{\pi}{4};0,0\right)| = |\cos\left(2\cdot\frac{\pi^2}{4^2}\right)-1| = 0.669252...$$

Definition:
Let Φ be a C^1 function, and $U, V \subset \mathbb{R}^n$ be open sets, with

$$\Phi : U \to V \quad \text{and} \quad u \mapsto \Phi(u)$$

Here, $u = (u_1, u_2, \cdots, u_n)^T$ and $\Phi(u) = (\Phi_1(u), \Phi_2(u), \cdots, \Phi_n(u))^T$.

The Jacobian matrix $J\Phi(u^0)$ is assumed to be regular for every $u^0 \in U$, and there exists a C^1 inverse function $\Phi^{-1} : V \to U$. Then, $x = \Phi(u)$ is referred to as a **coordinate transformation** from the coordinates u to the coordinates x.

Let $u = (r, \varphi)^T$
with $0 < r$ and $-\pi < \varphi < \pi$

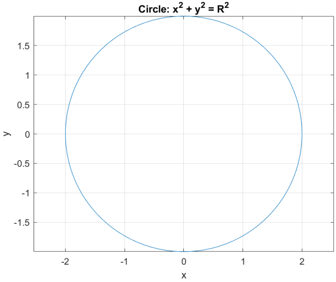
$$x = \begin{pmatrix} x \\ y \end{pmatrix} = \Phi(r, \varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \end{pmatrix}$$

Circle Equation

The equation for circle is:

$$x^2 + y^2 = R^2$$

describes the boundary K of a circular disk with a radius of R and a center at $(0, 0)$.
K can be represented using Polar coordinates with $R = r$.



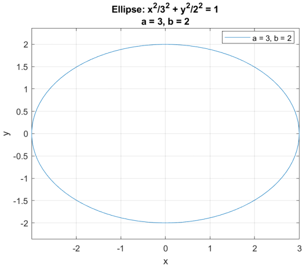
Ellipse Equation

The equation for ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

describes the boundary E of an ellipse with the semi-axes a and b and a center at $(0, 0)$.

E can be represented as $(x, y) = (a\cos(\varphi), b\sin(\varphi))$.



In cylindrical coordinates, a point is represented as $u = (r, \varphi, z)^T$ with $0 < r, -\pi < \varphi < \pi, z \in \mathbb{R}$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, z) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix}$$

In spherical coordinates, a point is represented as $u = (r, \varphi, \theta)^T$ with $0 < r, -\pi < \varphi < \pi, -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, \theta) = \begin{pmatrix} r\cos(\varphi)\sin(\theta) \\ r\sin(\varphi)\sin(\theta) \\ r\cos(\theta) \end{pmatrix}$$

The inequality

$$x^2 + y^2 + z^2 \leq R^2$$

describes a **Solid Sphere** K with a radius of R and a center at $(0, 0, 0)$.
With $0 \leq r \leq R$, K can be represented using spherical coordinates.

$$x^2 + y^2 = 3$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:

Circle: Radius $r = \sqrt{3}$, Center $(0, 0)$

Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (\sqrt{3}\cos(\varphi), \sqrt{3}\sin(\varphi))$$

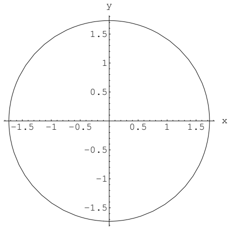


Figure: Circle $x^2 + y^2 = 3$

$$4x^2 + 9y^2 = 36$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:

Ellipse: Semi-axes $a = 3$ and $b = 2$, Center $(0, 0)$

Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (3\cos(\varphi), 2\sin(\varphi))$$

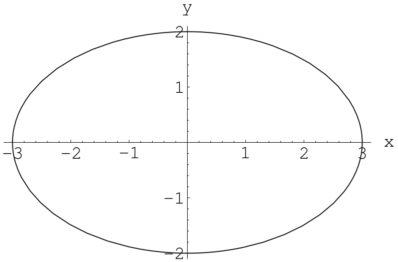


Figure: Ellipse $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$

Example 05

$$16x^2 + 3y^2 + 6y + 3 = 48$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:

Through completing the square, we obtain

$$16x^2 + 3y^2 + 6y + 3 = 16x^2 + 3(y+1)^2 = 48$$

$$\Leftrightarrow \quad \frac{x^2}{(\sqrt{3})^2} + \frac{(y+1)^2}{4^2} = 1$$

Ellipse: Semi-axes $a = \sqrt{3}$ and $b = 4$, Center $(0, -1)$

Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (\sqrt{3}\cos(\varphi), 4\sin(\varphi) - 1)$$

Example 05

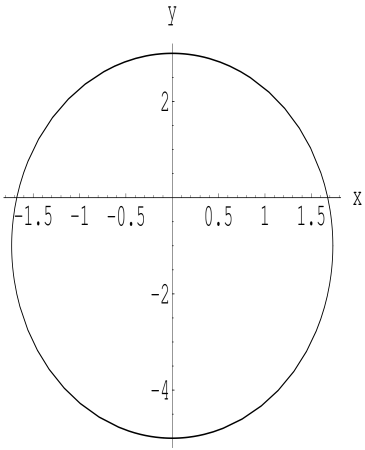


Figure: Ellipse $\dfrac{x^2}{(\sqrt{3})^2} + \dfrac{(y+1)^2}{4^2} = 1$

Example 06

$$x^2 - 6x + 9 + y^2 = 25$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:
Through completing the square, we obtain

$x^2 - 6x + 9 + y^2 = (x-3)^2 + y^2 = 5^2.$
Circle: Radius $r = 5$, Center $(3, 0)$

Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (5\cos(\varphi) + 3, 5\sin(\varphi))$$

Example 06

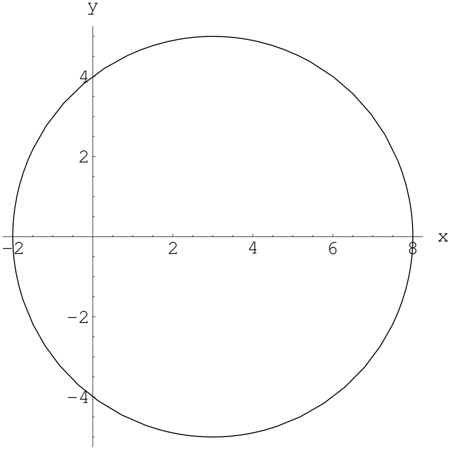


Figure: Circle $(x-3)^2 + y^2 = 5^2$

Draw the solution sets of the following region in $\mathbb{R}^3$ and represent them using cylindrical coordinates. $x^2 + y^2 \leq 4$ with $x \leq 0$ and $1 \leq z \leq 3$

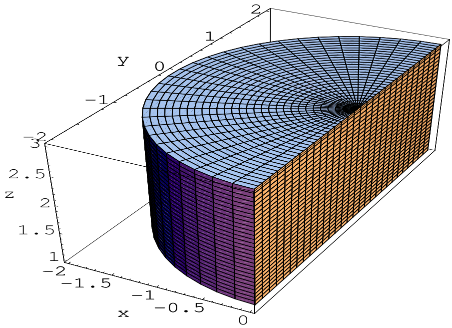


Figure: Half cylinder Z

Cylindrical Coordinates for Z: $\quad u = (r, \varphi, z)^T$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix} = \Phi(r, \varphi, z)$$

with $\quad 0 \le r \le 2\,, \quad \dfrac{\pi}{2} \le \varphi \le \dfrac{3\pi}{2}\,, \quad 1 \le z \le 3$

Draw the solution sets of the following region in $\mathbb{R}^3$ and represent them using spherical coordinates.$x^2 + y^2 + z^2 \leq 16$, $0 \leq y$

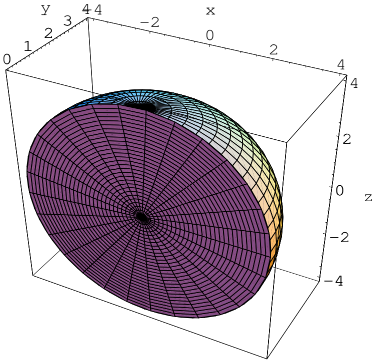


Figure: Half sphere H

Spherical Coordinates for H: $\quad u = (r, \varphi, \theta)^T$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, \theta) = \begin{pmatrix} r\cos(\varphi)\sin(\theta) \\ r\sin(\varphi)\sin(\theta) \\ r\cos(\theta) \end{pmatrix}$$

with $\quad 0 \leq r \leq 4\,, \quad 0 \leq \varphi \leq \pi\,, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Example 09

Consider the coordinate transformation

$$\Phi(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix}$$

with $(x,y) \in Q := [-1,1] \times [-1,1]$.

- Calculate $J\Phi(x,y)$ and $\det(J\Phi(x,y))$.
- Calculate $\Phi^{-1}(u,v), J\Phi^{-1}(u,v),\ \det(J\Phi^{-1}(u,v))$.
- Draw Q and $\Phi(Q)$

Example 09(i)

$$\Phi(x,y)=\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}=\begin{pmatrix} x-y \\ x+y \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

This is a linear transformation, more precisely, it's a rotation and scaling by 45° with a factor of $\sqrt{2}$, as:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}=\sqrt{2}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}=\sqrt{2}\begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix}.$$

$$J\Phi(x,y)=\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$\det(J\Phi(x,y))=2$$

$$\begin{aligned}\Phi^{-1}(u,v) &= \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \\ &= \begin{pmatrix} (u+v)/2 \\ (v-u)/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \\ J\Phi^{-1}(u,v) &= \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = (J\Phi)^{-1}, \end{aligned}$$

$$\det(J\Phi^{-1}(u,v)) = 1/2$$

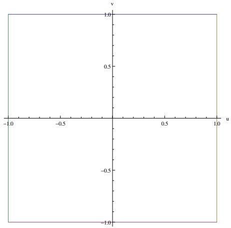


Figure: Q

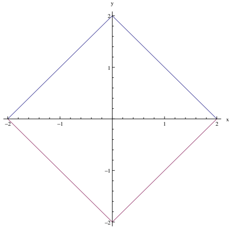


Figure: $\Phi(Q)$

THANK YOU