Analysis III: Auditorium Exercise-02

For Engineering Students

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Office Hours

- Mon 14:30-15:30 bi-weekly (21.10., 04.11., 18.11., 02.12., 16.12., 13.01.)
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Definition : Let D is open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$, $x^0 \in D$. The function f is called **Partially differentiable** in x^0 with respect to x_i, if the limit

$$\frac{\partial f}{\partial x_i}(x^0) := \lim_{t \to 0} \frac{f(x^0 + te_i) - f(x^0)}{t}$$

exists. Here e_i denotes the i-th unit vector and this limit is called partial derivative of f with respect to x_i at x^0.

Continuous Partial Derivatives:

If at every point x^0 the partial derivative with respect to every veriable $x_i, i = 1, ..., n$ exists and if the partial derivatives are continuous functions then we call f continuous partial differentiable or a C^1-function.

Definition : Let D is open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is partial differentiable then the gradient is

$$\text{grad } f(x_0) = \left(\frac{\partial f}{\partial x_1}(x^0), ..., \frac{\partial f}{\partial x_n}(x^0) \right)$$

The Symbolic vector

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n} \right)^T$$

and denote as a nabla-operator.
Thus we obtain the column vector

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x^0), ..., \frac{\partial f}{\partial x_n}(x^0) \right)^T$$

Example

Compute the gradient of the following functions:

- $f(x, y) = 2x^2 + 3y^2$
- $g(x, y) = 2x^2 + 3x^2y^2$
- $h(x, y) = 2cos(x) + 3e^y$

Definition :

- A contour line, which is also referred to as a level curve, represents the intersection of a specific surface with a horizontal plane defined by the equation $z = c$.
- When these contour lines are depicted together on the xy-plane, it forms a representation known as a contour map or contour plot.
- Offering valuable insights into the characteristics of the surface.

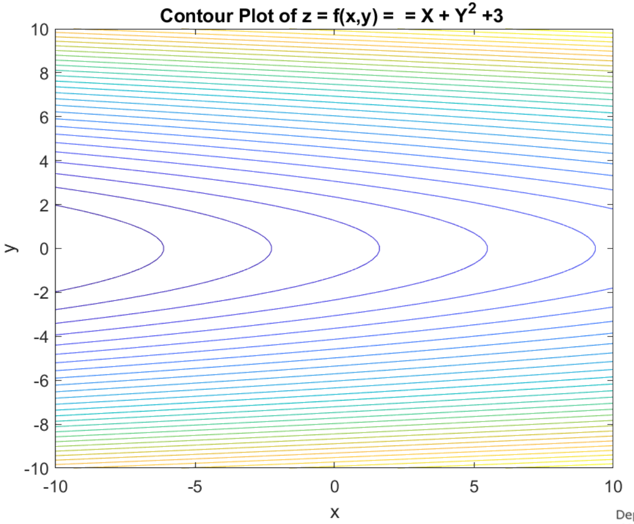
Plot contour lines of the given function

$$z = f(x, y) = 3 + x + y^2$$

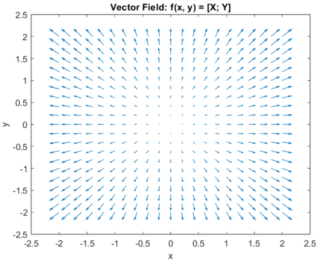
MATLAB code

```
x = linspace(-10, 10, 100);
y = linspace(-10, 10, 100);
[X, Y] = meshgrid(x, y);
F = X + Y.^2 + 3;
contour(X, Y, F, 30);
xlabel('x');
ylabel('y');
title('name');
```

Example



Let D be open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is called a **vector field** on D, if every function $f_i(x)$ of $f = (f_1, ..., f_n)^T$ is a $C^{\mathcal{K}}$-function, then f is called $c^{\mathcal{K}}$-vector field.



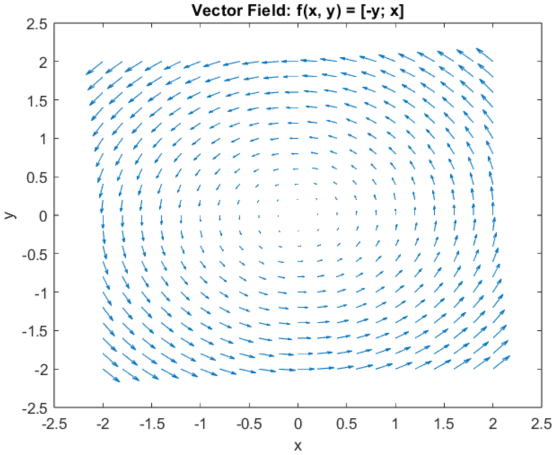
$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Example

Find the vector field in the xy -plane

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Example



Vector Field: f(x, y) = [-y; x]
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
y
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
x

Jacobian Matrix

Let $f : D \to \mathbb{R}^m$,$D \subset \mathbb{R}^n$, $x = (x_1, ..., x_n)^T \in D$,

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, ..., x_n) \\ f_2(x_1, ..., x_n) \\ \vdots \\ f_m(x_1, ..., x_n) \end{pmatrix}$$

Then the Jacobian Matrix is $m \times n$ matrix $J_{ij} = \frac{\partial f_i}{\partial x_j}(x)$:

$$Jf(x) = \begin{pmatrix} \text{grad } f_1(x) \\ \text{grad } f_2(x) \\ \vdots \\ \text{grad } f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \dots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) \dots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

A matrix is given

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

The determinant is

$$det\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right) = 2$$

- If $m = n$ the determinant of the Jacobian is knownn as the Jacobian determinant of f
- The Jacobian is used when making a change of variables and a coordinate transformation.

Compute the Jacobian matrix and the Jacobian determinant of the following vector function:

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x + x^2y \\ 5x + sin(y) \end{pmatrix}$$

Compute the Jacobian matrix of the following vector function:

$$f(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{pmatrix} = \begin{pmatrix} xe^y + x^2 z \\ e^{x^2+2y^2} \end{pmatrix}$$

Let $f : D \to \mathbb{R}^m$ be differentiable in $x^0 \in D$, D is open.
Let $g : E \to \mathbb{R}^k$ be differentiable in $y^0 = f(x^0 \in E \subset \mathbb{R}^m)$, E is open.
Then $g \circ f$ is differentiable in x^0.

For the differentials it holds

$$\mathrm{d}(g \circ f)(x^0) = \mathrm{d}g(y^0) \circ \mathrm{d}f(x^0)$$

and analogously for the Jacobian matrix

$$\mathrm{J}(g \circ f)(x^0) = \mathrm{J}g(y^0) \circ \mathrm{J}f(x^0)$$

Exercise 3

Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^3$ be a vector valued function of one variable defined as follows

$$f(x, y, z) = e^z cos(2x) sin(3y)$$

$$g(t) = (x(t), y(t), z(t)) = (2t, t^2, t^3)$$

Compute the derivative of the composition $f \circ g$.

Exercise 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^2 y + x y^2$$

and $g : \mathbb{R}^2 \to \mathbb{R}^2$

$$g(s, t) = \begin{pmatrix} x(s, t) \\ y(s, t) \end{pmatrix} = \begin{pmatrix} 2s + t \\ s - 2t \end{pmatrix}$$

Compute the derivative of the composition $f \circ g$.

Let $f : D \to \mathbb{R}$,$D \subset \mathbb{R}^n$, D is open , $x^0 \in D$, $v \in \mathbb{R}^n \backslash 0$ a vector. Then

$$D_v f(x^0) := \lim_{t \to 0^+} \frac{f(x_0 + tv) - f(x^0)}{t}$$

is called the directional derivative of $f(x)$ in the direction v.

$$g(x, y) = (1, 3x^2)^T$$

Exercise 5

Let $f(x, y) = x^2y$. Now compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(1, 2)$ at the point $(3, 2)$

Exercise 6

Let $f(x, y) = x^2y$. Now compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(2, 1)$ at the point $(3, 2)$

Exercise 7

Compute $D_v f(x,y)$ for $f(x,y) = cos(\frac{x}{y})$ in the direction $v = (3, -4)$

Exercise 8

Compute $D_v f(3,-1,0)$ for $f(x,y,z) = 4x - y^2 e^{3xz}$ in the direction $v = (-1,4,2)$. Is it a direction of descent or ascent?

For the vectorfield $= (f_1, \ldots, f_n)^T$ in $\mathbb{R}^n$ and $x = (x_1, ...x_n)$ then the divergence of the vector field is

$$\text{div } f := \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

- The divergence of the vector field f is a scalar field.

Example

Compute the divergence of the vector field:

- $\vec{f} = x\hat{i} + y\hat{j}$
- $\vec{f} = -x\hat{i} - y\hat{j}$
- $\vec{f} = -y\hat{i} + x\hat{j}$

Let f be the in $\mathbb{R}^3$ and the rotation of f or the curl of f:

$$f(x_1,x_2,x_3) = \begin{pmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{pmatrix}, \quad \text{rot } f := \begin{pmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \\ \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{pmatrix}$$

- The curl of a vector field $\vec{f}$ is indeed a vector field. The curl operation takes a vector field as input and produces another vector field as its output.

Example

Compute the divergence of the vector field:

- $\vec{f} = x\hat{i} + y\hat{j}$
- $\vec{f} = -y\hat{i} + x\hat{j}$

Exercise 9

Compute the div (f) and rot (f) for $f(x, y, z) = \begin{pmatrix} x^2y \\ z^3 - 3x \\ 4y^2 \end{pmatrix}$

Exercise 10

Compute the div (f) and rot (f) for $f(x, y, z) = 2x^2 z\vec{i} + yz\vec{j} + \vec{k}$.

Additional Exercise

Compute the Jacobian matrix of the following vector function:

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{pmatrix} = \begin{pmatrix} sin(y) \\ x^3 + cos(x) \\ x^2 y^2 \end{pmatrix}$$

THANK YOU