Mathematics Department
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WiSe 2024/2025

Mathematik III Exam
(Module: Analysis III)

March 4, 2025

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Exercise	Points	Evaluater
1		
2		
3		
4		

Exercise 1: (7 Points)

Determine and classify all local extrema of $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = 8x - \frac{9}{2}y$$

under the constraint

$$g(x,y) = 16x^2 + 9y^2 - 25 = 0$$

using the Lagrange multiplier rule. First check the regularity condition.

Solution:

Regularity condition:

$\operatorname{grad} g(x,y) = \binom{32x}{18y} = \binom{0}{0} \iff \binom{x}{y} = \binom{0}{0}$

The regularity condition is satisfied on the admissible set, since $\binom{0}{0}$ is not an admissible point.. **[1 Point]**

Hence a necessary condition for the (local) optimality is

$\operatorname{grad} F(x,y) = \operatorname{grad}\,(f(x,y) + \lambda g(x,y)) = \mathbf{0}$.

We have to solve the system of equations

$$\begin{aligned} f_x + \lambda g_x &= 8 + \lambda \cdot 32x = 0, & (\Longrightarrow \lambda \neq 0) \\ f_y + \lambda g_y &= -\frac{9}{2} + \lambda \cdot 18y = 0 & \\ g(x,y) &= 16x^2 + 9y^2 - 25 = 0\,. & [\,\mathbf{1\ Point}] \end{aligned}$$

λ cannot be equal to 0. We obtain

I : $8 + 2 \cdot 16\lambda x = 0 \Longrightarrow x = -\frac{1}{4\lambda}$

II : $-\frac{9}{2} + 2 \cdot 9\lambda y = 0 \Longrightarrow y = \frac{1}{4\lambda}$.

Hence $y = -x$. Inserting this into the third equation results in

$g(x,x) = 16x^2 + 9x^2 - 25 \overset{!}{=} 0 \Longrightarrow x^2 = 1$.

This gives two candidates for local extrema:

$$P_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Computing the candidates : **(3 Points)**

Classification [**2 Points**]

The boundary of an ellipse is a compact set, therefore a comparison of the function values is sufficient for classifikation.

$f(-1,1) = -8 - \frac{9}{2} = -\frac{25}{2}$, $\qquad f(1,-1) = 8 + \frac{9}{2} = \frac{25}{2}$.

In P_1 we have a (the global) minimum and in P_2 a (the global) maximum.

Alternatively one can calculate the Hessian

$$\boldsymbol{H}\ \boldsymbol{F}\,(x,y) = \begin{pmatrix} 32\lambda & 0 \\ 0 & 18\lambda \end{pmatrix}$$

For P_1 we calculate $\lambda_1 = \frac{1}{4}$ and

$$\boldsymbol{H}\ \boldsymbol{F}\,(-1,-1) = \begin{pmatrix} 8 & 0 \\ 0 & \frac{9}{2} \end{pmatrix} \qquad \text{(positive definite)}$$

Here we have a minimum.

For P_2 with $\lambda_2 = -\frac{1}{4}$ one calculates

$$\boldsymbol{H}\ \boldsymbol{F}\,(1,1) = \begin{pmatrix} -8 & 0 \\ 0 & -\frac{9}{2} \end{pmatrix} \qquad \text{(negative definite)}$$

Hence we have a maximum.

Exercise 2: (4 Points)

Given the half circular ring

$$D := \left\{(x,y)^T \in \mathbb{R}^2;\ 1 \leq x^2 + y^2 \leq 9,\ y \geq 0\right\}$$

with mass density $\rho(x,y) = 3 - y$

compute the mass m of D.

Exercise:

Using polar coordinates

$x = r\cos(\phi),\ y = r\sin(\phi),\ r \in [1,3],\ \phi \in [0,\pi]$ **(1 Point)**

we calculate the masse m

$$\begin{aligned}
m &= \int_1^3 \int_0^\pi \rho(x(r,\phi), y(r,\phi))\, r\, d\phi dr \\
&= \int_1^3 \int_0^\pi (3 - r\sin(\phi))\, r\, d\phi dr \qquad \textbf{(1Punkt)} \\
&= \int_1^3 \Big[3r\phi + r^2\cos(\phi)\Big]_0^\pi\, dr = \int_1^3 \Big(3r(\pi - 0) + r^2(-1-1)\Big)\, dr \\
&= 3\pi \left[\frac{r^2}{2}\right]_1^3 - \left[\frac{2r^3}{3}\right]_1^3 = \frac{3\pi}{2}(9-1) - \left(\frac{2}{3}(27-1)\right) \qquad \textbf{(2 Points)} \\
&= 12\pi - \frac{52}{3}.
\end{aligned}$$

Exercise 3: (5 Points)

a) Show that there exists no potential for the function $g : \mathbb{R}^3 \to \mathbb{R}^3$

$$g(x,y,z) \;=\; (x + \frac{1}{2}yz\,,\; -y+z\,,\; -y).$$

b) Compute the line integral

$$\int_c g(x,y,z)d(x,y,z)$$

along the curve

$$c(t) \;=\; \begin{pmatrix} t \\ \sin(\frac{t}{2}) \\ \cos(\frac{t}{2}) \end{pmatrix} \qquad c \;:\; [0\,,\,\pi] \to \mathbb{R}^3.$$

Solution:

a) For example because of $(g_1)_z \neq 0 = (g_3)_x$ the Jacobian is not symmetric, hence there exists no potential for g. **(1 Point)**

b) **(4 Points)** $\dot{\boldsymbol{c}}\,(t) = \begin{pmatrix} 1 \\ \frac{1}{2}\cos(\frac{t}{2}) \\ -\frac{1}{2}\sin(\frac{t}{2}) \end{pmatrix}, \qquad \boldsymbol{g}\,(\,\boldsymbol{c}\,(t)) = \begin{pmatrix} t + \frac{1}{2}\sin(\frac{t}{2})\cos(\frac{t}{2}) \\ -\sin(\frac{t}{2}) + \cos(\frac{t}{2}) \\ -\sin(\frac{t}{2}) \end{pmatrix}.$

$$\begin{aligned} <\, g(\boldsymbol{c}(t)), \dot{\boldsymbol{c}}(t)\, > \;&=\; t + \frac{1}{2}\sin(\frac{t}{2})\cos(\frac{t}{2}) \\ &+ \frac{1}{2}\cos(\frac{t}{2})\left(-\sin(\frac{t}{2}) + \cos(\frac{t}{2})\right) \\ &- \frac{1}{2}\sin(\frac{t}{2})\left(-\sin(\frac{t}{2})\right) = t + \frac{1}{2}\left(\cos^2(\frac{t}{2}) + \sin^2(\frac{t}{2})\right) \\ &= t + \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \int_{\boldsymbol{c}} g(x,y,z)d(x,y,z) \;&=\; \int_0^\pi \,<\, g(\boldsymbol{c}(t)), \dot{\boldsymbol{c}}(t)\, >\, dt \\ &= \int_0^\pi t + \frac{1}{2}\, dt = \\ &= \frac{t^2}{2} + \frac{t}{2}\bigg|_0^\pi = \frac{\pi^2 + \pi}{2}\,. \end{aligned}$$

Exercise 4: (1+3 Points)

Let $\boldsymbol{f}$ be the vector field

$$\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2, \ \boldsymbol{f}(x,y) = \begin{pmatrix} xy + \tan(e^{-x^2}) \\ x^2 - \cos(e^{-y^2}) \end{pmatrix}$$

a) Compute $\mathbf{curl}\,\boldsymbol{f}(x,y)$.

b) For the mathematically positively oriented boundary ∂D of the triangle

$$D := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,:\, 0 \le x \le 2,\, 0 \le y \le 2 - x \right\}$$

compute the line Integral $\displaystyle\int_{\partial D} \boldsymbol{f}(x,y)\, d(x,y)$.

Solution:

a)

$$\mathbf{curl}\ \boldsymbol{f}(x,y) = (f_2)_x - (f_1)_y = 2x - x = x. \qquad \textbf{(1 Point)}$$

b) Green's theorem gives:

$$I := \int_{\partial D} \boldsymbol{f}(x,y)\, d(x,y) = \int_D \mathrm{rot} f(x,y)\, d(x,y) \qquad \textbf{(Ansatz: 1 Point)}$$

Hence

$$I = \int_0^2 \int_0^{2-x} x\, dy\, dx = \int_0^2 x\, [y]_0^{2-x}\, dx \qquad \textbf{(1 Point)}$$

$$= \int_0^2 (2x - x^2)\, dx = \left[x^2 - \frac{x^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}. \qquad \textbf{(1 Point)}$$