Mathematics Department
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SuSe 2025

Mathematik III Exam
(Module: Analysis III)
August 27, 2025

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Exercise	Points	Evaluater
1		
2		
3		
4		

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Exercise 1: (4 Points)

Determine and classify the stationary point of the function $f : \mathbb{R}^2 \to \mathbb{R}$

$$\boldsymbol{f}\,(x, y) := x^2 - 4xy + 36y^2 - 10x - 12y\,.$$

Solution:

$$\begin{aligned} f_x(x,y) &= 2x - 4y - 10 = 0 \iff x = 2y + 5. \\ f_y(x,y) &= -4x + 72y - 12 = -8y - 20 + 72y - 12 = 0 \iff 64y = 32 \\ &\iff y = \frac{1}{2} \implies x = 6. \qquad \textbf{[2 points]} \end{aligned}$$

For the Hessian matrix one computes: $H(x, y) = \begin{pmatrix} 2 & -4 \\ -4 & 72 \end{pmatrix}$. [**1 point**]

The main subdeterminants of the Hessian matrix are positive

$H_{11} = 2 > 0$ and $\det(H) = 144 - 16 > 0$.

Alternatively: compute the eigenvalues:

$(2 - \lambda)(72 - \lambda) - 16 = 0 \iff \lambda^2 - 74\lambda + 144 - 16 = \lambda^2 - 74\lambda + 128 = 0$

$\lambda_{1,2} = 37 \pm \underbrace{\sqrt{37^2 - 128}}_{<37} > 0.$

In $P = (6, \frac{1}{2})$ the function f has a (local) minimum. [**1 point**]

Exercise 2: (3+3 Points)

a) Determine a potential for the function $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$\boldsymbol{f}\,(x, y, z) \;=\; (2xy^2\,,\; z + 2yx^2\,,\; y)^T.$$

b) Compute the line integral

$$\int_{\boldsymbol{c}} \boldsymbol{f}\,(x, y, z) d(x, y, z)$$

along the curve

$$\boldsymbol{c}\,(t) \;=\; \begin{pmatrix} t+1 \\ t^2+2 \\ t^3 \end{pmatrix} \qquad \boldsymbol{c} \;:\; [0\,,\,1] \to \mathbb{R}^3.$$

Solution:

a) A potential Φ satisfies

$$\Phi_x \;=\; 2xy^2\,, \quad \Phi_y \;=\; z + 2yx^2\,, \quad \Phi_z \;=\; y.$$

$$\Phi_x \;=\; 2xy^2 \quad \Longleftrightarrow \quad \Phi(x, y, z) \;=\; x^2y^2 \;+\; c(y, z)$$

$$\Phi_y \;=\; 2yx^2 \;+\; c_y(y, z) \;\stackrel{!}{=}\; z + 2yx^2 \quad \Longleftrightarrow \quad c_y(y, z) \;=\; z$$

$$\Longleftrightarrow\; c(y, z) \;=\; yz \;+\; d(z) \;\Longrightarrow\; \Phi(x, y, z) \;=\; x^2y^2 \;+\; yz + d(z)$$

$$\Phi_z \;=\; y + d'(z) \;\stackrel{!}{=}\; y \;\Longrightarrow\; d'(z) = 0 \;\Longrightarrow\; d = Konst.$$

Hence $\Phi(x, y, z) \;=\; x^2y^2 \;+\; yz$ is a potential for f.

b) With

$$\boldsymbol{c}\,(0) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\,,\; \boldsymbol{c}\,(1) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

and $\Phi(\,\boldsymbol{c}\,(0)) \;=\; 4\,,$

as well as $\Phi(\,\boldsymbol{c}\,(1)) \;=\; 2^2 \cdot 3^2 + 3 \;=\; 39$

we easily compute

$$\int_{\boldsymbol{c}} \boldsymbol{f}(x, y, z) d(x, y, z) \;=\; \Phi(\boldsymbol{c}(1)) \;-\; \Phi(\boldsymbol{c}(0))$$

$$=\; 39 - 4 \;=\; 35.$$

Exercise 3: (5+1+3+1 Points)

Consider the half ball $K := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,:\, x^2 + y^2 + z^2 \leq 25\,,\ z \geq 0 \right\}$

and the vector field $\boldsymbol{f}\,(x, y, z) = \begin{pmatrix} z^2 + x^2 \\ x^2 + y \\ 2z + 1 \end{pmatrix}$.

a) Compute the integral $\displaystyle\int_K \operatorname{div}\ \boldsymbol{f}\,(x, y, z)\, d(x, y, z)\,.$

Note: Depending on the order of the variables you integrate by, the following **might** be helpful

$$2\cos^2(\alpha) = \cos(2\alpha) + 1.$$

b) The domain K is bounded by a flat surface B and a curved surface M. Determine a parametrization of the flat surface B.

c) Compute the flow of $\boldsymbol{f}$ through B.

d) Determine the flow of $\boldsymbol{f}$ through M using the results from parts a) and c).

Solution:

a) $\operatorname{div}\ \boldsymbol{f}\,(x, y, z) = 2x + 1 + 2 = 2x + 3\,.$ **(1 point)**

Parametrization of K using spherical coordinates:

$x = r\cos(\phi)\cos(\theta)\,, \quad y = r\sin(\phi)\cos(\theta)\,, \quad z = r\sin(\theta)\,,$
$0 \leq r \leq 5,\ 0 \leq \phi \leq 2\pi,\ 0 \leq \theta \leq \frac{\pi}{2}\,.$ **(1 point)**

$$\int_K \operatorname{div} \boldsymbol{f}(x, y, z)\, d(x, y, z)$$

$$= \int_0^5 \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (2r\cos(\phi)\cos(\theta) + 3) \cdot r^2 \cos(\theta)\, d\phi\, d\theta\, dr \qquad \textbf{(1 point)}$$

$$= \int_0^5 \int_0^{\frac{\pi}{2}} \left[2r^3\cos^2(\theta)\sin(\phi) + 3r^2\cos(\theta)\phi\right]_0^{2\pi}\, d\theta\, dr$$

$$= \int_0^5 \int_0^{\frac{\pi}{2}} 6\pi r^2 \cos(\theta)\, d\theta\, dr = \int_0^5 6\pi r^2 \left[\sin(\theta)\right]_0^{\frac{\pi}{2}}\, dr$$

$$= 2\pi \int_0^5 3r^2\, dr = \left[2\pi r^3\right]_0^5 = 250\pi. \qquad \textbf{(2 points)}$$

b) The domain K is bounded by a flat surface B with parametrization:

$$p(r, \phi) := \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \\ 0 \end{pmatrix}, \qquad r \in [0, 5], \quad \phi \in [0, 2\pi], \qquad \textbf{(1 point)}$$

and the upper half of the surface of a sphere M.

c) **[3 points]**

For the flux through B we first calculate:

$$\frac{\partial p}{\partial r} = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix} \qquad \frac{\partial p}{\partial \phi} = \begin{pmatrix} -r\sin(\phi) \\ r\cos(\phi) \\ 0 \end{pmatrix}$$

$$\frac{\partial p}{\partial r} \times \frac{\partial p}{\partial \phi} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \qquad f(p(r,\phi)) = \begin{pmatrix} \text{irrelevant} \\ \text{irrelevant} \\ 1 \end{pmatrix}$$

$$< f, \frac{\partial p}{\partial \phi} \times \frac{\partial p}{\partial r} >= -r\ .$$

Hence we obtain the flux

$$\int_0^5 \int_0^{2\pi} < f, \frac{\partial p}{\partial \phi} \times \frac{\partial p}{\partial r} > d\phi dr = \int_0^5 \int_0^{2\pi} -r\, d\phi dr$$

$$= \int_0^5 -2\pi r dr = -25\pi\,.$$

d) Following Gauss theorem and using a) - c) we have

flux through the surface of K = flux through B + flux through M = $\int_K \operatorname{div}(x,y,z)\, d(x,y,z)$

For the flux through the curved surface M we obtain

$250\pi + 25\pi = 275\pi.$ **(1 point)**