Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Work Sheet 7, Solutions

Exercise 1:

Given a vector field $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\boldsymbol{f}(x,y,z) = \left(\sin y + 3x^2z^2, x\cos y + \frac{1}{1+y^2}, 1 + 2x^3z\right)^T .$$

a) Show the existence of a potential for $\boldsymbol{f}$ without calculating it.

b) Calculate a potential by successively integrating $\boldsymbol{f}$ and

c) using the fundamental theorem for line integrals.

d) Given a curve $\mathbf{c} : [0, 3\pi/2] \to \mathbb{R}^3$ with $\mathbf{c}(t) = (\cos t, 0, \sin t)^T$. Compute the line integral

$$\int_c \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} .$$

e) Plot the curve $\mathbf{c}$ using the MATLAB function 'plot3'.

Solution:

a) $\mathbb{R}^3$ is simply connected and the integrability condition

$$\text{rot}\,\boldsymbol{f}(x,y,z) = \begin{pmatrix} f_{3y} - f_{2z} \\ f_{1z} - f_{3x} \\ f_{2x} - f_{1y} \end{pmatrix} = \begin{pmatrix} 0-0 \\ 6x^2z - 6x^2z \\ \cos y - \cos y \end{pmatrix} = \mathbf{0}$$

is fulfilled. Hence there exist a potential $v(x,y,z)$ for $\boldsymbol{f}(x,y,z)$, i.e. it holds $\boldsymbol{f} = \text{grad}\ v = (v_x, v_y, v_z)$.

b) $v_x(x,y,z) = \sin y + 3x^2z^2 \quad \Rightarrow \quad v(x,y,z) = x\sin y + x^3z^2 + c(y,z)$

$$\Rightarrow \quad v_y(x,y,z) = x\cos y + c_y(y,z) \stackrel{!}{=} x\cos y + \frac{1}{1+y^2}$$

$$\Rightarrow \quad c_y(y,z) = \frac{1}{1+y^2} \quad \Rightarrow \quad c(y,z) = \arctan y + k(z)$$

$$\Rightarrow \quad v(x,y,z) = x\sin y + x^3z^2 + \arctan y + k(z)$$

$$\Rightarrow \quad v_z(x,y,z) = 2x^3z + k'(z) \stackrel{!}{=} 1 + 2x^3z$$

$$\Rightarrow \quad k'(z) = 1 \quad \Rightarrow \quad k(z) = z + K \quad \text{with } K \in \mathbb{R}$$

$$\Rightarrow \quad v(x,y,z) = x\sin y + x^3z^2 + \arctan y + z + K$$

c) Choose as a curve $\boldsymbol{k}$ the line connecting points $(0,0,0)$ and (x,y,z), i.e. $\boldsymbol{k}(t) = t(x,y,z)^T$ with $0 \le t \le 1$, so the potential v of $\boldsymbol{f}$ can be calculated according to the fundamental theorem for line integrals

$$\begin{aligned}
v(x,y,z) &= \int_{\boldsymbol{k}} \boldsymbol{f}(\boldsymbol{x})\,d\boldsymbol{x} + K = \int_0^1 \boldsymbol{f}(\boldsymbol{k}(t))\dot{\boldsymbol{k}}(t)\,dt + K \\
&= \int_0^1 \left\langle \begin{pmatrix} \sin(ty) + 3(tx)^2(tz)^2 \\ tx\cos(ty) + \dfrac{1}{1+(ty)^2} \\ 1 + 2(tx)^3tz \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle dt + K \\
&= \int_0^1 5x^3z^2t^4 + x\sin(ty) + txy\cos(ty) + \frac{y}{1+(ty)^2} + z\,dt + K \\
&= x^3z^2t^5 + tx\sin(ty) + \arctan(ty) + zt\Big|_0^1 + K \\
&= x^3z^2 + x\sin(y) + \arctan(y) + z + K
\end{aligned}$$

d) Since the potential $v(x,y,z)$ for $\boldsymbol{f}(x,y,z)$ exists, from the fundamental theorem for line integrals it follows

$$\int_c \boldsymbol{f}(\boldsymbol{x})\,d\boldsymbol{x} = v(\mathbf{c}(3\pi/2)) - v(\mathbf{c}(0)) = v(0,0,-1) - v(1,0,0) = -1$$

e)

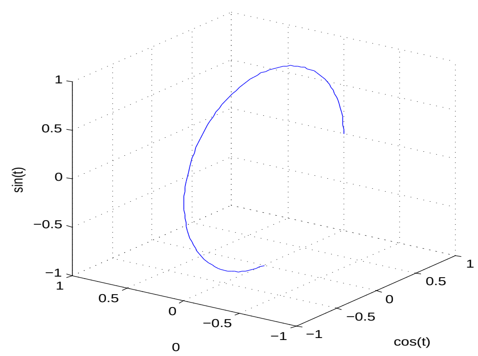

Figure 1 Curve $\mathbf{c}$

Exercise 2:

Given a vector field $\boldsymbol{f}\,(x, y, z) = \left(0, 0, z^3\right)^T$ and the body

$$H = \left\{(x, y, z)^T \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 \leq 16\,,\; 0 \leq y \right\}\,.$$

a) Make a sketch of H.

b) Give parameterizations for each of surface segments bounding H.

c) Calculate the flow of $\boldsymbol{f}$ through these boundary segments.

d) Compute the volume integral $\displaystyle\int_H \operatorname{div}\,\boldsymbol{f}\,(x, y, z)\, d(x, y, z)\,.$

Solution:

a)

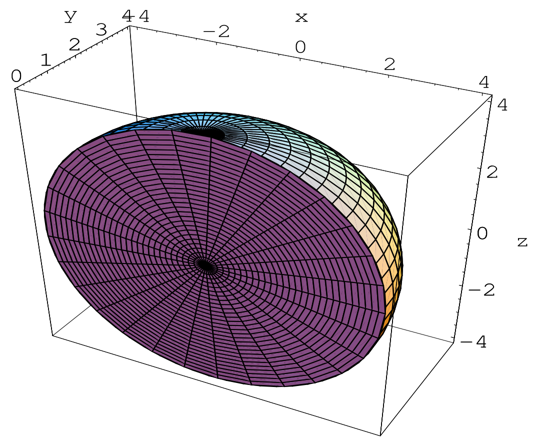

Figure 2: Hemisphere H

b) Parameterization of the circular face S: $\boldsymbol{p}\,:[0, 4] \times [0, 2\pi] \to \mathbb{R}^3$ with

$$\boldsymbol{p}\,(r, \varphi) = \begin{pmatrix} r\cos\varphi \\ 0 \\ r\sin\varphi \end{pmatrix}$$

Parameterization of the hemisphere T: $\boldsymbol{q}\,:[0, \pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3$ with

$$\boldsymbol{q}\,(\varphi, \psi) = \begin{pmatrix} 4\cos\varphi\cos\psi \\ 4\sin\varphi\cos\psi \\ 4\sin\psi \end{pmatrix}$$

c) The flow through S, with outer normal vectors

$$\frac{\partial \boldsymbol{p}}{\partial r} \times \frac{\partial \boldsymbol{p}}{\partial \varphi} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \cos\varphi & 0 & \sin\varphi \\ -r\sin\varphi & 0 & r\cos\varphi \end{vmatrix} = \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}$$

$$\int_S \boldsymbol{f}\ do = \int_0^4 \int_0^{2\pi} \left\langle \begin{pmatrix} 0 \\ 0 \\ r^3 \sin^3\varphi \end{pmatrix}, \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix} \right\rangle d\varphi dr = \int_0^4 \int_0^{2\pi} 0\, d\varphi dr = 0$$

Flow through T, with the outer normal vectors

$$\frac{\partial \boldsymbol{q}}{\partial \varphi} \times \frac{\partial \boldsymbol{q}}{\partial \psi} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ -4\sin\varphi\cos\psi & 4\cos\varphi\cos\psi & 0 \\ -4\cos\varphi\sin\psi & -4\sin\varphi\sin\psi & 4\cos\psi \end{vmatrix} = 16\cos\psi \begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ \sin\psi \end{pmatrix}$$

$$\int_T \boldsymbol{f}\ do = \int_0^{\pi} \int_{-\pi/2}^{\pi/2} 16\cos\psi \left\langle \begin{pmatrix} 0 \\ 0 \\ 4^3\sin^3\psi \end{pmatrix}, \begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ \sin\psi \end{pmatrix} \right\rangle d\psi d\varphi$$

$$= \int_0^{\pi} \int_{-\pi/2}^{\pi/2} 4^5 \cos\psi \sin^4\psi d\psi d\varphi = 4^5\pi\, \frac{\sin^5\psi}{5}\Big|_{-\pi/2}^{\pi/2} = \frac{2\cdot 4^5\pi}{5}$$

d) Using the Gauss's theorem (divergence theorem), one obtains:

$$\int_H \operatorname{div}\ \boldsymbol{f}\ d(x,y,z) = \int_S \boldsymbol{f}\ do + \int_T \boldsymbol{f}\ do = \frac{2\cdot 4^5\pi}{5}$$

Alternatively: direct calculation using spherical coordinates:

$$\begin{aligned} &\int_H \operatorname{div}\ \boldsymbol{f}(x,y,z)\, d(x,y,z) \\ = &\int_H 3z^2\, d(x,y,z) = \int_0^4 \int_0^{\pi} \int_{-\pi/2}^{\pi/2} 3r^2\sin^2\psi r^2\cos\psi\, d\psi d\varphi dr \\ = &\int_0^4 r^4 dr \int_0^{\pi} d\varphi \int_{-\pi/2}^{\pi/2} 3\cos\psi\sin^2\psi\, d\psi = \frac{r^5}{5}\Big|_0^4 \cdot \varphi|_0^{\pi} \cdot \sin^3\psi\Big|_{-\pi/2}^{\pi/2} = \frac{2\cdot 4^5\pi}{5} \end{aligned}$$

Discussion: 29.1. - 2.2.2024