Department of Mathematics, University of Hamburg **WiSe 2023/24**
Prof. Dr. J. Struckmeier
Dr. K. Rothe, Md. T. Hassan

Analysis III for Engineering Students

Homework sheet 7, Solutions

Exercise 1:

Verify Green's theorem for the vector field

$$\boldsymbol{f}(x,y) = (x^2 + y, \sin x)^T$$

and the area G enclosed by the function $y = 1 - (x-1)^2$ and the x axis.

Solution:

$$G := \left\{ (x,y)^T \in \mathbb{R}^2 \mid 0 \le x \le 2\,,\ 0 \le y \le 1 - (x-1)^2 \right\}$$

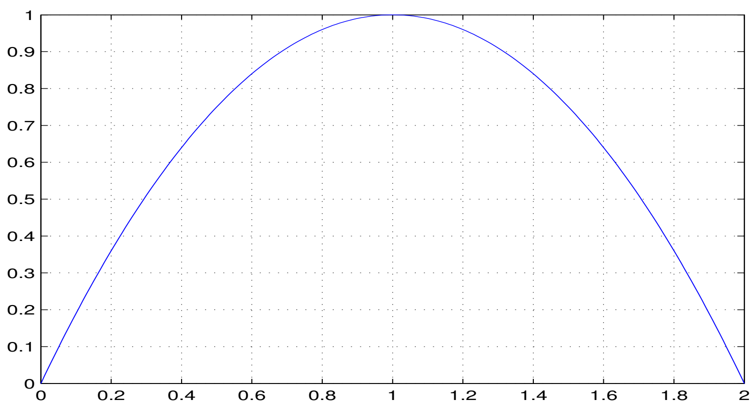

Figure 1: Area G

Parameterization of the boundary of G by:

$$\mathbf{c}_1(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}, \quad \mathbf{c}_2(t) = \begin{pmatrix} t \\ 1-(t-1)^2 \end{pmatrix}, \quad t \in [0,2] \quad \Rightarrow \quad \partial G = \mathbf{c}_1 - \mathbf{c}_2\,.$$

$$\begin{aligned}
\oint_{\partial G} \boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x} &= \int_{\mathbf{c}_1} \boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x} - \int_{\mathbf{c}_2} \boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x} \\
&= \int_0^2 < \boldsymbol{f}(\mathbf{c}_1(t)), \dot{\mathbf{c}}_1(t) > dt - \int_0^2 < \boldsymbol{f}(\mathbf{c}_2(t)), \dot{\mathbf{c}}_2(t) > dt \\
&= \int_0^2 \left\langle \begin{pmatrix} t^2 \\ \sin t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle dt \\
&\quad - \int_0^2 \left\langle \begin{pmatrix} t^2 + 1 - (1-t)^2 \\ \sin t \end{pmatrix}, \begin{pmatrix} 1 \\ -2(t-1) \end{pmatrix} \right\rangle dt \\
&= \int_0^2 t^2 - (t^2 + 1 - (1 - 2t + t^2) - 2(t-1)\sin t)\, dt \\
&= \int_0^2 t^2 - 2t + 2(t-1)\sin t\, dt \\
&= \frac{t^3}{3} - t^2 - 2(t-1)\cos t + 2\sin t \Big|_0^2 \\
&= \frac{8}{3} - 4 - 2\cos 2 + 2\sin 2 - 2 \;=\; 2\sin 2 - 2\cos 2 - \frac{10}{3}
\end{aligned}$$

$$\begin{aligned}
\int_G \text{rot } \boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x} &= \int_0^2 \int_0^{1-(x-1)^2} (\sin x)_x - (x^2 + y)_y \, dydx \;=\; \int_0^2 \int_0^{1-(x-1)^2} \cos x - 1 \, dydx \\
&= \int_0^2 (1 - (x-1)^2)(\cos x - 1)\, dx \\
&= \int_0^2 \cos x - 1 + (x-1)^2 - (x-1)^2 \cos x \, dx \\
&= \sin x - x + \frac{(x-1)^3}{3} - (x-1)^2 \sin x - 2(x-1)\cos x + 2\sin x \Big|_0^2 \\
&= \sin 2 - 2 + \frac{1}{3} - \sin 2 - 2\cos 2 + 2\sin 2 - (-\frac{1}{3} + 2) \\
&= 2\sin 2 - 2\cos 2 - \frac{10}{3}
\end{aligned}$$

Green's theorem (divergence theorem): $\oint_{\partial G} \boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x} = 2\sin 2 - 2\cos 2 - \frac{10}{3} = \int_G \text{rot } \boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x}$

Exercise 2:

Given the saddle area

$$S = \left\{ (x,y,z)^T \in \mathbb{R}^3 \mid x^2 + y^2 \le 4\,,\ z = xy \right\}\ ,$$

a) derive a parameterization of S,

b) plot S using the MATLAB function 'ezgraph3' and

c) calculate the area of S using a surface integral.

Solution:

a) We describe two parameterization variants:

(i) Parameterization in Cartesian coordinates via the function graph

$$\boldsymbol{p}\,(x,y) = (x,y,xy)^T \quad \text{with} \quad x^2 + y^2 \le 4,$$

(ii) Parameterization using polar coordinates

$$\boldsymbol{q}\,(r,\varphi) = (r\cos\varphi, r\sin\varphi, r^2\cos\varphi\sin\varphi))^T \quad \text{with} \quad (r,\varphi) \in [0,2]\times[0,2\pi]\,.$$

b) For the drawing with MATLAB we use the parameterization $\boldsymbol{q}$

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ezgraph3('surf','r*cos(s)','r*sin(s)','r^2*cos(s)*sin(s)',[0,2,0,2*pi])
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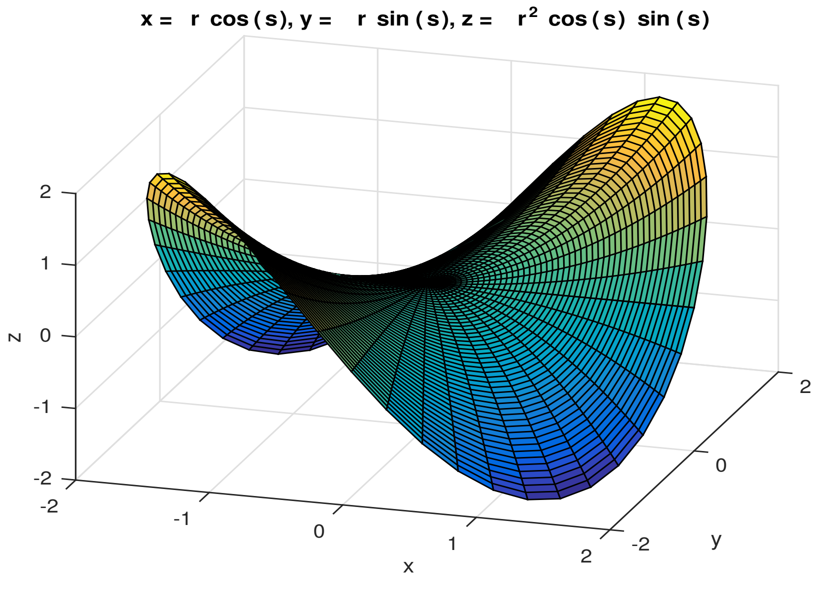

Figure 2: Saddle surface S

c) The calculation of the surface of S is simpler by using the parameterization $\boldsymbol{p}$ and then switching to polar coordinates using the transformation theorem.

$$\frac{\partial \boldsymbol{p}}{\partial x} \times \frac{\partial \boldsymbol{p}}{\partial y} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix} \Rightarrow \left\| \frac{\partial \boldsymbol{p}}{\partial x} \times \frac{\partial \boldsymbol{p}}{\partial y} \right\| = \sqrt{1+x^2+y^2}$$

$$\begin{aligned} \int_S do &= \int_{x^2+y^2\le 4} \left\| \frac{\partial \boldsymbol{p}}{\partial x} \times \frac{\partial \boldsymbol{p}}{\partial y} \right\| d(x,y) = \int_{x^2+y^2\le 4} \sqrt{1+x^2+y^2}\, d(x,y) \\ &= \int_0^2 \int_0^{2\pi} r\sqrt{1+r^2}\, d\varphi dr = 2\pi \int_0^2 r\sqrt{1+r^2}\, dr \\ &= \frac{2\pi}{3} \left(1+r^2)^{3/2}\right)\Big|_0^2 = \frac{2\pi(5^{3/2}-1)}{3} = 21.3217... \end{aligned}$$

The alternative solution using the parameterization $\boldsymbol{q}$:

$$\frac{\partial \boldsymbol{q}}{\partial r} \times \frac{\partial \boldsymbol{q}}{\partial \varphi} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \cos\varphi & \sin\varphi & 2r\cos\varphi\sin\varphi \\ -r\sin\varphi & r\cos\varphi & r^2(\cos^2\varphi - \sin^2\varphi) \end{vmatrix} = r \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \cos\varphi & \sin\varphi & 2r\cos\varphi\sin\varphi \\ -\sin\varphi & \cos\varphi & r(1-2\sin^2\varphi) \end{vmatrix}$$

$$= r \begin{pmatrix} r\sin\varphi(1-2\sin^2\varphi) - 2r\cos^2\varphi\sin\varphi \\ -(r\cos\varphi(1-2\sin^2\varphi) + 2r\cos\varphi\sin^2\varphi) \\ 1 \end{pmatrix} = r \begin{pmatrix} -r\sin\varphi \\ -r\cos\varphi \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \left\| \frac{\partial \boldsymbol{q}}{\partial r} \times \frac{\partial \boldsymbol{q}}{\partial \varphi} \right\| = \left\| r \begin{pmatrix} -r\sin\varphi \\ -r\cos\varphi \\ 1 \end{pmatrix} \right\| = r\sqrt{1+r^2}$$

$$\Rightarrow \quad \int_S do = \int_0^2 \int_0^{2\pi} \left\| \frac{\partial \boldsymbol{q}}{\partial r} \times \frac{\partial \boldsymbol{q}}{\partial \varphi} \right\| d\varphi\, dr = \int_0^2 \int_0^{2\pi} r\sqrt{1+r^2}\, d\varphi\, dr$$

Submission deadline: 2.2.2024