Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Work Sheet 6, Solutions

Exercise 1:

Compute the following integrals:

a) $\displaystyle\int_0^1 \int_0^2 (2x+y)^2 \, dy \, dx \,,$

b) $\displaystyle\int_R \frac{1}{xy^2+x} \, d(x,y) \quad \text{with} \quad R = [1,2] \times [0,1] \,,$

c) $\displaystyle\int_Q \cos y + y\sqrt{x+z} \, d(x,y,z) \quad \text{with} \quad Q = [0,2] \times [0,\pi] \times [1,2] \,.$

Solution:

a)
$$\begin{aligned}\int_0^1 \int_0^2 (2x+y)^2 \, dy \, dx &= \int_0^1 \left.\frac{(2x+y)^3}{3}\right|_0^2 dx = \frac{1}{3}\int_0^1 8(x+1)^3 - 8x^3 \, dx \\ &= \frac{2}{3}\left((x+1)^4 - x^4\right)\Big|_0^1 = \frac{2}{3}(16-1-1) = \frac{28}{3}\end{aligned}$$

b)
$$\begin{aligned}\int_R \frac{1}{xy^2+x} \, d(x,y) &= \int_0^1 \int_1^2 \frac{1}{x} \cdot \frac{1}{y^2+1} \, dx \, dy = \int_0^1 \frac{1}{y^2+1} \left(\int_1^2 \frac{1}{x} \, dx \right) dy \\ &= \left(\int_1^2 \frac{1}{x} \, dx \right) \cdot \left(\int_0^1 \frac{1}{y^2+1} \, dy \right) \\ &= \left(\ln|x| \big|_1^2 \right) \cdot \left(\arctan y \big|_0^1 \right) = \frac{\pi \ln 2}{4}\end{aligned}$$

c) $$\int_Q \cos y + y\sqrt{x+z}\, d(x,y,z) = \int_1^2\int_0^2\int_0^\pi \cos y + y\sqrt{x+z}\, dy\, dx\, dz$$
$$= \int_1^2\int_0^2 \left(\sin y + \frac{y^2}{2}\sqrt{x+z}\right)\Big|_0^\pi\, dx\, dz$$
$$= \frac{\pi^2}{2}\int_1^2\int_0^2 \sqrt{x+z}\, dx\, dz$$
$$= \frac{\pi^2}{3}\int_1^2 \left((x+z)^{3/2}\right)\Big|_0^2\, dz = \frac{\pi^2}{3}\int_1^2 (2+z)^{3/2} - z^{3/2}\, dz$$
$$= \frac{2\pi^2}{15}\left((2+z)^{5/2} - z^{5/2}\right)\Big|_1^2 = \frac{2\pi^2}{15}\left(33 - 4\sqrt{2} - 9\sqrt{3}\right)$$

Exercise 2:

a) Draw the closed area K given by $x \le 0,\ y \le 0,\ 0 \le z$ and $x^2+y^2+z^2 = 9$ and represent it as a "normal" area.

b) Compute $\displaystyle\int_K 8yz\, d(x,y,z)$.

Solution:

a) $x \le 0,\ y \le 0,\ 0 \le z$ describes an octant in $\mathbb{R}^3$ and
$x^2+y^2+z^2 = 9$ is a spherical surface of radius $r = 3$.

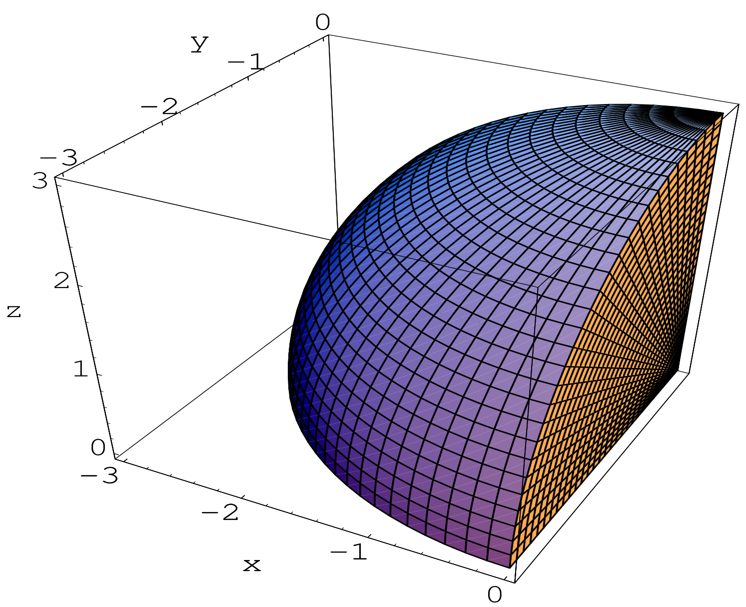

Figure 2 1/8 sphere K

$$K = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} -3 \le x \le 0\,, \\ -\sqrt{9-x^2} \le y \le 0\,, \\ 0 \le z \le \sqrt{9-x^2-y^2} \end{array} \right\}$$

b)

$$\begin{aligned}
\int_K 8yz\, d(x,y,z) &= \int_{-3}^{0} \int_{-\sqrt{9-x^2}}^{0} \int_{0}^{\sqrt{9-x^2-y^2}} 8yz\, dz\, dy\, dx \\
&= \int_{-3}^{0} \int_{-\sqrt{9-x^2}}^{0} 4yz^2\Big|_0^{\sqrt{9-x^2-y^2}}\, dy\, dx \\
&= \int_{-3}^{0} \int_{-\sqrt{9-x^2}}^{0} 4y(9-x^2-y^2)\, dy\, dx \\
&= \int_{-3}^{0} 2y^2(9-x^2) - y^4\Big|_{-\sqrt{9-x^2}}^{0}\, dx \\
&= \int_{-3}^{0} -(9-x^2)^2\, dx = -\frac{648}{5}
\end{aligned}$$

or alternatively with transformation to spherical coordinates:

$$\begin{aligned}
\int_K 8yz\, d(x,y,z) &= \int_{0}^{\pi/2} \int_{\pi}^{3\pi/2} \int_{0}^{3} 8(r\sin(\varphi)\cos(\psi))(r\sin(\psi))r^2\cos(\psi)\, dr\, d\varphi\, d\psi \\
&= 8 \int_0^3 r^4\, dr \int_{\pi}^{3\pi/2} \sin(\varphi)\, d\varphi \int_0^{\pi/2} \sin(\psi)\cos^2(\psi)\, d\psi \\
&= 8 \left(\frac{r^5}{5}\Big|_0^3 \right) \left(-\cos(\varphi)\big|_{\pi}^{3\pi/2} \right) \left(-\frac{\cos^3(\psi)}{3}\Big|_0^{\pi/2} \right) \\
&= \frac{8\cdot 3^5}{5} \cdot (-1) \cdot \frac{1}{3} = -\frac{648}{5}
\end{aligned}$$

Exercise 3:

Given a rotational paraboloid P by $x^2 + y^2 \leq 4$ and $0 \leq z \leq 4 - x^2 - y^2$. P has a constant density ρ .

a) Plot P using the MATLAB-function 'ezgraph3'.

b) For P compute the mass and moment of inertia with respect to the z axis.

c) Compute the moment of inertia of P with respect to the axis D , parallel to the z axis, passing through the point $(1, 1, 5)^T$.

Solution

a) The MATLAB plot command is

```
ezgraph3('surf','r*cos(t)','r*sin(t)','4-r^2',[0,2,0,2*pi])
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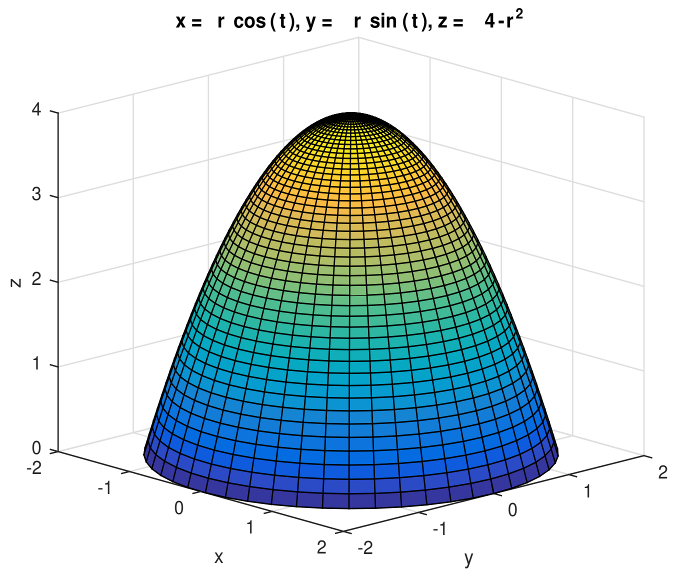

Figure 3 Rotational paraboloid P .

b) Computation of the mass M in cylindrical coordinates using the transformation theorem with constant density ρ :

$$\begin{aligned} M &= \int_P \rho \, d(x,y,z) = \rho \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r \, dz \, d\varphi \, dr = \rho \int_0^2 \int_0^{2\pi} r z\Big|_0^{4-r^2} \, d\varphi \, dr \\ &= \rho \int_0^2 \int_0^{2\pi} r(4-r^2) \, d\varphi \, dr = \rho \int_0^2 r(4-r^2)\varphi\Big|_0^{2\pi} \, dr \\ &= 2\pi\rho \int_0^2 r(4-r^2) \, dr = 8\pi\rho \end{aligned}$$

Calculation of the moment of inertia with respect to the z axis in cylindrical coordinates using the transformation theorem with the constant density ρ

$$\begin{aligned}
\Theta_{z\text{-axis}} &= \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} \rho r^2 r \, dz \, d\varphi \, dr \;=\; \rho \int_0^2 \int_0^{2\pi} r^3 z\Big|_0^{4-r^2} \, d\varphi \, dr \\
&= \rho \int_0^2 \int_0^{2\pi} r^3(4-r^2) \, d\varphi \, dr \;=\; \rho \int_0^2 r^3(4-r^2)\varphi\Big|_0^{2\pi} \, dr \\
&= 2\pi\rho \int_0^2 r^3(4-r^2) \, dr \;=\; \frac{32\pi\rho}{3}
\end{aligned}$$

c) For reasons of symmetry, the center of mass of P is on the z axis. This is confirmed mathematically using transformation to cylindrical coordinates

$$\begin{aligned}
x_s &= \frac{1}{M} \int_Z \rho x \, d(x,y,z) \;=\; \frac{\rho}{M} \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r\cos(\varphi) r \, dz \, d\varphi \, dr \\
&= \frac{\rho}{M} \int_0^2 \int_0^{4-r^2} r^2 \, dz \, dr \int_0^{2\pi} \cos(\varphi) \, d\varphi = 0 \\
y_s &= \frac{1}{M} \int_Z \rho y \, d(x,y,z) \;=\; \frac{\rho}{M} \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r\sin(\varphi) r \, dz \, d\varphi \, dr \\
&= \frac{\rho}{M} \int_0^2 \int_0^{4-r^2} r^2 \, dz \, dr \int_0^{2\pi} \sin(\varphi) \, d\varphi = 0
\end{aligned}$$

Therefore, according to Steiner's theorem, it holds

$$\Theta_D = Md^2 + \Theta_{z\text{-axis}} = 8\pi\rho(1^2+1^2) + \frac{32\pi\rho}{3} = \frac{80\pi\rho}{3} \,.$$

Discussion: 15.1. - 19.1.2024