Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Homework sheet 6, Solutions

Exercise 1:

a) For the function

$$f : Q \to \mathbb{R}\,, \quad f(x,y) = 6 - 2x + 4y$$

with $Q := [0,3] \times [0,2]$ compute

(i) Riemannian upper and lower sum for the following equidistant decomposition Z of Q

$$Q_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]\,, \quad i,j = 1,\ldots,n$$

where $x_i = \dfrac{3i}{n}$ and $y_j = \dfrac{2j}{n}$

(ii) and the integral of f over Q using Fubini's theorem.

b) (i) Draw the area P enclosed by the functions $f(x) = 2x$ and $g(x) = 24 - 2x^2$ and represent it as the "normal" area.

(ii) Compute $\displaystyle\int_P x \, d(x,y)$.

Solution:

a) (i)

$$\begin{aligned}
U_f(Z) &= \sum_{i,j=1}^{n} \inf_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \mathrm{Vol}(Q_{i,j}) \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (6 - 2x_i + 4y_{j-1}) \cdot \frac{3}{n} \cdot \frac{2}{n} \right) \\
&= \frac{6}{n^3} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (6n - 6i + 8(j-1)) \right) \\
&= \frac{6}{n^3} \sum_{i=1}^{n} \left(6n^2 - 6in + 8\frac{(n-1)n}{2} \right) \\
&= \frac{6}{n^3} \left(6n^3 - 6n\frac{n(n+1)}{2} + 4n^2(n-1) \right) = 42\left(1 - \frac{1}{n}\right)
\end{aligned}$$

$$O_f(Z) = \sum_{i,j=1}^{n} \sup_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \mathrm{Vol}(Q_{i,j})$$
$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (6 - 2x_{i-1} + 4y_j) \cdot \frac{3}{n} \cdot \frac{2}{n} \right) = 42\left(1 + \frac{1}{n}\right)$$

(ii) $$\int_Q 6 - 2x + 4y \, d(x,y) = \int_0^3 \left(\int_0^2 6 - 2x + 4y \, dy \right) dx = \int_0^3 6y - 2xy + 2y^2 \Big|_0^2 \, dx$$
$$= \int_0^3 12 - 4x + 8 \, dx = 20x - 2x^2 \Big|_0^3 = 42$$

So we obtain:

$$42\left(1 - \frac{1}{n}\right) = U_f(Z) \leq \int_Q f(x,y)\, d(x,y) = 42 \leq O_f(Z) = 42\left(1 + \frac{1}{n}\right).$$

b) (i) Intersections of f and g:

$$2x = 24 - 2x^2 \Leftrightarrow 0 = x^2 + x - 12 = (x+4)(x-3) \Leftrightarrow x_1 = -4\,,\ x_2 = 3.$$

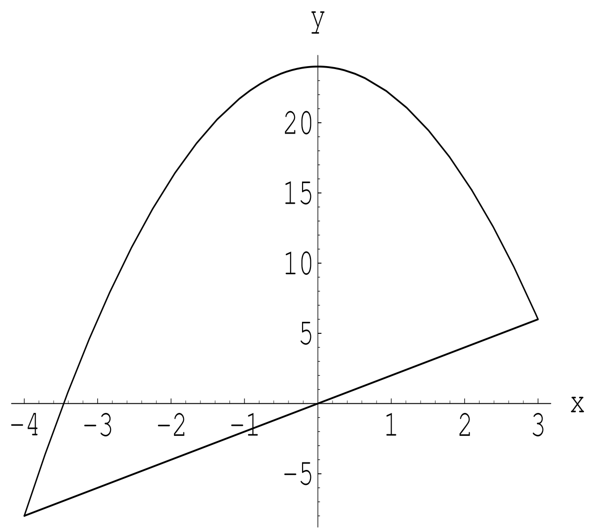

Figure 1 “normal” area P

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; -4 \leq x \leq 3\,,\ 2x \leq y \leq 24 - 2x^2 \right\}$$

(ii) $$\int_P x \, d(x,y) = \int_{-4}^{3} \int_{2x}^{24-2x^2} x \, dy \, dx = \int_{-4}^{3} (xy)\Big|_{2x}^{24-2x^2} \, dx$$
$$= \int_{-4}^{3} x(24 - 2x^2 - 2x)\, dx = -\frac{343}{6}$$

Exercise 2:

Draw the half cylinder Z given by $1 \leq z \leq 2$, $0 \leq y$ and $x^2 + y^2 \leq 9$ and calculate its center of mass with the density function $\rho(x, y, z) = z$ using cylindrical coordinates.

Solution:

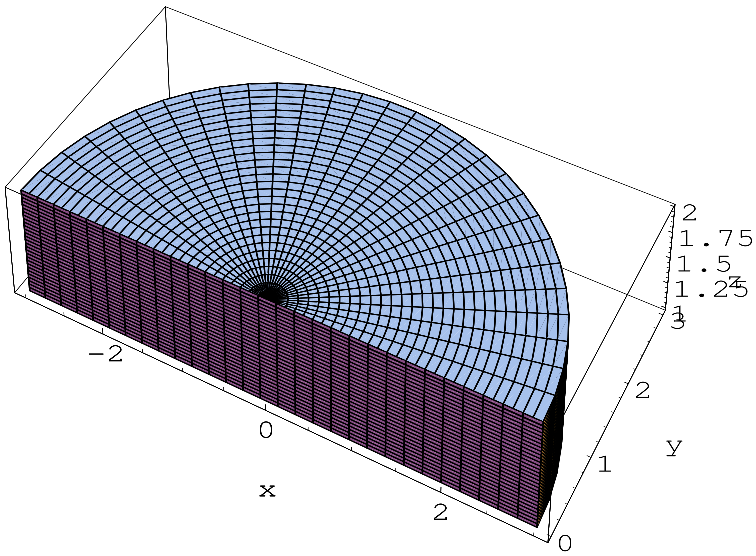

Figure 2 half cylinder Z

Cylindrical coordinates for Z: $0 \leq r \leq 3$, $0 \leq \varphi \leq \pi$, $1 \leq z \leq 2$ with

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix} = \mathbf{\Phi}(r, \varphi, z), \quad \det \boldsymbol{J}\,\mathbf{\Phi}(r, \varphi, z) = r$$

Calculation of the mass M in cylindrical coordinates using the transformation theorem with $\rho(x, y, z) = z$:

$$\begin{aligned} M &= \int_Z z\, d(x, y, z) = \int_0^3 \int_0^\pi \int_1^2 zr\, dz\, d\varphi\, dr = \int_0^3 \int_0^\pi \left. \frac{rz^2}{2} \right|_1^2 d\varphi\, dr \\ &= \frac{3}{2} \int_0^3 \int_0^\pi r\, d\varphi\, dr = \frac{3}{2} \int_0^3 \pi r\, dr = \frac{27\pi}{4} \end{aligned}$$

Computation of the center of mass (x_s, y_s, z_s):

$$\begin{aligned} x_s &= \frac{1}{M} \int_Z zx\, d(x, y, z) = \frac{1}{M} \int_0^3 \int_0^\pi \int_1^2 zr \cdot r\cos(\varphi)\, dz\, d\varphi\, dr \\ &= \frac{3}{2M} \int_0^3 \int_0^\pi r^2 \cos(\varphi)\, d\varphi\, dr = \frac{3}{2M} \int_0^3 r^2 \sin(\varphi)\Big|_0^\pi dr = 0 \end{aligned}$$

$x_s = 0$ is also due to the symmetry.

$$\begin{aligned} y_s &= \frac{1}{M}\int\limits_Z zy\, d(x,y,z) = \frac{1}{M}\int\limits_0^3\int\limits_0^\pi\int\limits_1^2 zr\cdot r\sin(\varphi)\, dz\, d\varphi\, dr \\ &= \frac{3}{2M}\int\limits_0^3\int\limits_0^\pi r^2\sin(\varphi)\, d\varphi\, dr = \frac{3}{2M}\int\limits_0^3 -r^2\cos(\varphi)\Big|_0^\pi\, dr \\ &= \frac{3}{M}\int\limits_0^3 r^2\, dr = \frac{27}{M} = \frac{4}{\pi} \end{aligned}$$

$$\begin{aligned} z_s &= \frac{1}{M}\int\limits_Z z^2\, d(x,y,z) = \frac{1}{M}\int\limits_0^3\int\limits_0^\pi\int\limits_1^2 z^2 r\, dz\, d\varphi\, dr \\ &= \frac{7}{3M}\int\limits_0^3\int\limits_0^\pi r\, d\varphi\, dr = \frac{7\pi}{3M}\int\limits_0^3 r\, dr = \frac{21\pi}{2M} = \frac{14}{9} \end{aligned}$$

Exercise 3:

a) For the vector field $\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2$ with $\boldsymbol{f}(x,y) = \begin{pmatrix} y + \sin x \\ xy^2 \end{pmatrix}$ calculate the integral of the curve (line integral) $\oint_{\mathbf{c}} \boldsymbol{f}(\boldsymbol{x}) d\boldsymbol{x}$.

Here $\mathbf{c}$ is the mathematically positive boundary curve of the area G enclosed by $x^2 \le y \le x$ with $0 \le x \le 1$.

b) For the vector field $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ with $\boldsymbol{f}(x,y,z) = \begin{pmatrix} -z^2/2 \\ 0 \\ xz \end{pmatrix}$

calculate the line integral $\int_{\mathbf{c}} \boldsymbol{f}(\boldsymbol{x}) d\boldsymbol{x}$ with the line

$$\mathbf{c} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3 \quad \text{and} \quad \mathbf{c}(t) = \begin{pmatrix} 2\cos^2 t \\ 2\sin t \cos t \\ 2\sin t \end{pmatrix} .$$

Solution:

a) The boundary curve consists of two smooth parts: $\partial G = \mathbf{c}_1 + \mathbf{c}_2$, with

$$\mathbf{c}_1(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix} , \quad \mathbf{c}_2(t) = \begin{pmatrix} 1-t \\ 1-t \end{pmatrix} , \quad \text{with} \quad 0 \le t \le 1$$

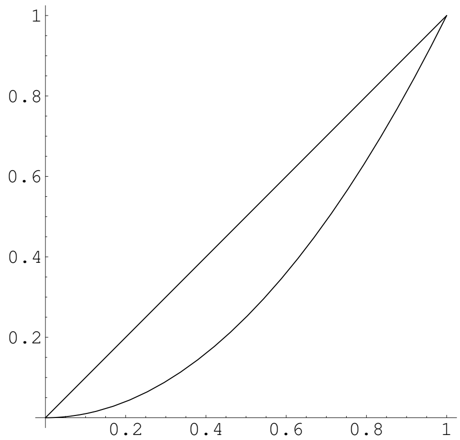

Figure 3 a) boundary curve ∂G

To calculate the line integral of the 2nd type, the tangential vectors are required:

$$\dot{\mathbf{c}}_1(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} , \quad \dot{\mathbf{c}}_2(t) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} .$$

$$\begin{aligned} \oint_{\partial G} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} &= \int_{\mathbf{c}_1} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\mathbf{c}_2} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int_0^1 \langle \boldsymbol{f}(\mathbf{c}_1(t)), \dot{\mathbf{c}}_1(t) \rangle \, dt + \int_0^1 \langle \boldsymbol{f}(\mathbf{c}_2(t)), \dot{\mathbf{c}}_2(t) \rangle \, dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left\langle \begin{pmatrix} t^2+\sin t \\ t\cdot t^4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2t \end{pmatrix} \right\rangle dt \\
&\quad + \int_0^1 \left\langle \begin{pmatrix} 1-t+\sin(1-t) \\ (1-t)\cdot(1-t)^2 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\rangle dt \\
&= \int_0^1 t^2+\sin t+2t^6+t-1+\sin(t-1)+(t-1)^3\,dt \\
&= \frac{t^3}{3}-\cos t+\frac{2t^7}{7}+\frac{(t-1)^2}{2}-\cos(t-1)+\frac{(t-1)^4}{4}\Big|_0^1 = -\frac{11}{84}
\end{aligned}
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b)

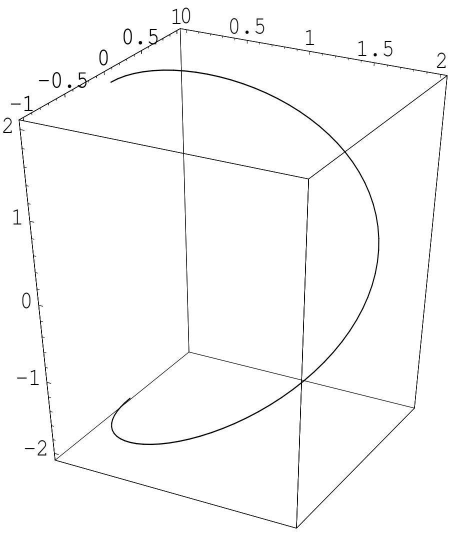

Figure 3 b) Curve **c**

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\begin{aligned}
\int_{\mathbf{c}} \boldsymbol{f}(\boldsymbol{x})\,d\boldsymbol{x} &= \int_{-\pi/2}^{\pi/2} \langle \boldsymbol{f}(\mathbf{c}(t)), \dot{\mathbf{c}}(t)\rangle\ dt \\
&= \int_{-\pi/2}^{\pi/2} \left\langle \begin{pmatrix} -2\sin^2 t \\ 0 \\ 4\sin t\cos^2 t \end{pmatrix}, \begin{pmatrix} -4\sin t\cos t \\ 2(\cos^2 t-\sin^2 t) \\ 2\cos t \end{pmatrix} \right\rangle dt \\
&= \int_{-\pi/2}^{\pi/2} 8\sin^3 t\cos t+8\sin t\cos^3 t\,dt \\
&= \int_{-\pi/2}^{\pi/2} 8\sin t\cos t\,dt \;=\; 4\sin^2 t\Big|_{-\pi/2}^{\pi/2} \;=\; 0
\end{aligned}
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Submission deadline: 19.1.2024