Department of Mathematics, University of Hamburg **WiSe 2023/24**
Prof. Dr. J. Struckmeier
Dr. K. Rothe, Md. T. Hassan

Analysis III for Engineering Students

Work Sheet 5, Solutions

Exercise 1:

Examine the implicitly given by the level set curve(s)

$$f(x, y) := y^4 - 2y^2 + x^4 - 2x^2 = 0.$$

In particular, determine

a) the symmetries of the curve(s),

b) the points of the curve with the horizontal and

c) vertical tangents,

d) the singular points of the curve and classify them,

e) draw the level set.

Solution:

$$f(x, y) := y^4 - 2y^2 + x^4 - 2x^2 = 0\,, \qquad \text{grad } f(x, y) = (4x(x^2 - 1), 4y(y^2 - 1))^T$$

a) The curve can have the following symmetries:
about the x-axis, $f(x, y) = f(x, -y)$,
about the y-axis, $f(x, y) = f(-x, y)$,
about the origin, $f(x, y) = f(-x, -y)$,
with respect to the bisecting line, $f(x, y) = f(y, x)$

b) We obtain the points of the curve with a horizontal tangent from the conditions

$$f_x(x, y) = 0 \quad \wedge \quad f(x, y) = 0 \quad \wedge \quad f_y(x, y) \neq 0\,.$$

$0 = f_x(x, y) = 4x(x^2 - 1) \quad \Rightarrow$

1. case: $x = 0$

$\Rightarrow \quad 0 = f(0,y) = y^2(y^2-2) \quad \Rightarrow \quad y = 0 \vee y = \pm\sqrt{2}$

$$\Rightarrow \quad P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; P_1 = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}, \; P_2 = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}.$$

2. case: $x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1$

$\Rightarrow \quad 0 = f(\pm 1, y) = y^4 - 2y^2 - 1 = (y^2-1)^2 - 2$

$\Rightarrow \quad y = \pm\sqrt{1+\sqrt{2}} \quad \Rightarrow \quad P_3 = \begin{pmatrix} 1 \\ \sqrt{1+\sqrt{2}} \end{pmatrix},$

$$P_4 = \begin{pmatrix} 1 \\ -\sqrt{1+\sqrt{2}} \end{pmatrix}, \; P_5 = \begin{pmatrix} -1 \\ \sqrt{1+\sqrt{2}} \end{pmatrix}, \; P_6 = \begin{pmatrix} -1 \\ -\sqrt{1+\sqrt{2}} \end{pmatrix}.$$

Since the condition $f_y(P_i) \neq 0$ is only fulfilled for $P_1, \cdots, P_6$, these are the points with horizontal tangent.

c) We obtain the points of the curve with a vertical tangent from the conditions

$$f_y(x,y) = 0 \quad \wedge \quad f(x,y) = 0 \quad \wedge \quad f_x(x,y) \neq 0\,.$$

$0 = f_y(x,y) = 4y(y^2-1) \quad \Rightarrow \quad y = 0 \quad \vee \quad y^2 - 1 = 0$

$y = 0 \quad \Rightarrow \quad 0 = f(x,0) = x^2(x^2-2) \quad \Rightarrow \quad x = 0 \vee x = \pm\sqrt{2}$

$$\Rightarrow \quad P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; P_7 = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}, \; P_8 = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}.$$

$y^2 - 1 = 0 \quad \Rightarrow \quad y = \pm 1$

$\Rightarrow \quad 0 = f(x, \pm 1) = x^4 - 2x^2 - 1 = (x^2-1)^2 - 2$

$\Rightarrow \quad x = \pm\sqrt{1+\sqrt{2}} \quad \Rightarrow \quad P_9 = \begin{pmatrix} \sqrt{1+\sqrt{2}} \\ 1 \end{pmatrix},$

$$P_{10} = \begin{pmatrix} -\sqrt{1+\sqrt{2}} \\ 1 \end{pmatrix}, \; P_{11} = \begin{pmatrix} \sqrt{1+\sqrt{2}} \\ -1 \end{pmatrix}, \; P_{12} = \begin{pmatrix} -\sqrt{1+\sqrt{2}} \\ -1 \end{pmatrix}.$$

The points $P_7, \cdots, P_{12}$ are the only ones to fulfill the condition $f_x(P_i) \neq 0$. Hence only for them we have vertical tangents. This result can also be obtained using the symmetries.

d) For $P_0 = (0,0)^T$ it holds grad $f(0,0) = \mathbf{0}$, hence P_0 is the only singular point.

$$\boldsymbol{H}\, f(x,y) = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 12y^2 - 4 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{H}\, f(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

It is an isolated point (maximum) because $\det \boldsymbol{H}\, f(0,0) > 0$ holds.

e)

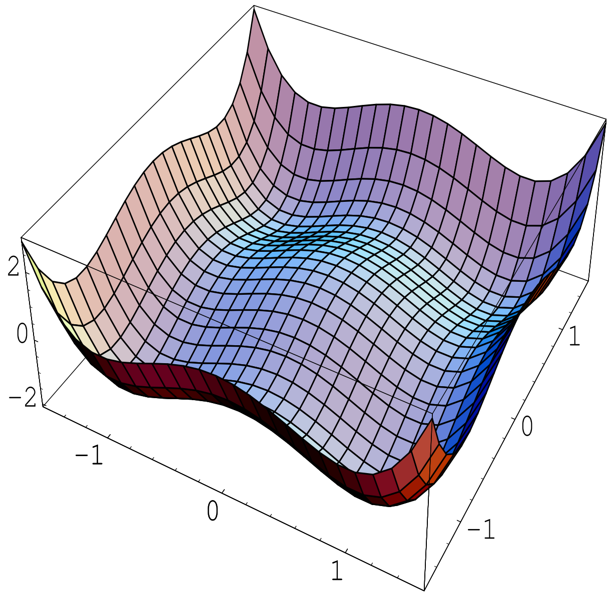

Figure 1 a) $f(x,y) = y^4 - 2y^2 + x^4 - 2x^2$

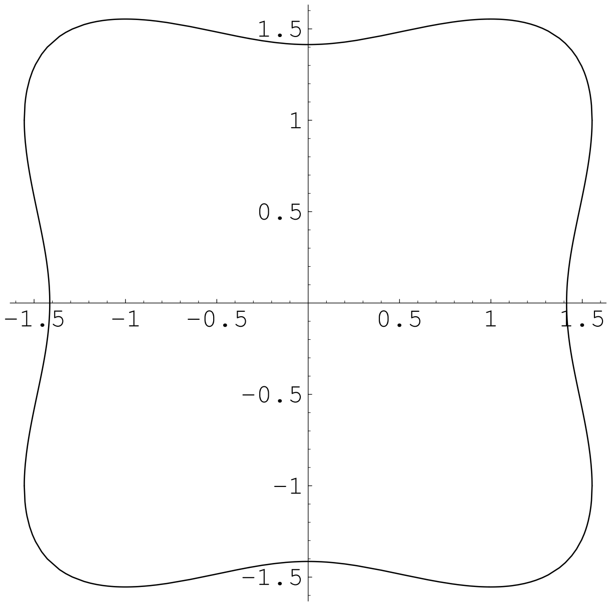

Figure 1 b) Level set $y^4 - 2y^2 + x^4 - 2x^2 = 0$, which contains the isolated point $P_0 = (0,0)^T$

Exercise 2:

Compute and classify the extrema of the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = 4x^2 + y^2$ on the circle $x^2 + y^2 - 2x = 3$

a) by using Lagrange multipliers method and

b) by using polar coordinate parameterization $\mathbf{c}$ of the circle and then solving the extreme problem $h(t) := f(\mathbf{c}(t))$.

Solution:

We need to deteremine the extrema of the function $f(x,y) = 4x^2 + y^2$ under the constraint $g(x,y) := (x-1)^2 + y^2 - 4 = 0$.

a) Regularity condition:

$\boldsymbol{J}\, g(x,y) = \text{grad } g(x,y)^T = (2(x-1), 2y) = (0,0) \Rightarrow (x,y) = (1,0)$

It holds $g(1,0) = -4$, i.e. the center of the circle $(1,0)$ does not belong to the circle $g = 0$. Hence all admissible points, i.e. $g(x,y) = 0$ satisfy the regularity condition

$$\text{Rang}\,(\boldsymbol{J}\, g(x,y)) = 1\,.$$

Lagrange-function: $\quad F(x,y) = 4x^2 + y^2 + \lambda((x-1)^2 + y^2 - 4)$

Lagrange multipliers rule:

$$\begin{pmatrix} \nabla F(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 8x + 2\lambda(x-1) \\ 2y(1+\lambda) \\ (x-1)^2 + y^2 - 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2. Equation:

1.case: $y = 0 \Rightarrow 0 = g(x,0) = (x-1)^2 - 4 = 0 \Rightarrow x_1 = 3\,,\ x_2 = -1$

Candidates for extrema: $\quad P_1 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}\,,\ P_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

2.case: $\lambda = -1 \Rightarrow 0 = 8x - 2(x-1) = 6x + 2 \Rightarrow x_3 = -1/3$

$\Rightarrow 0 = g(-1/3, y) = (-1/3 - 1)^2 + y^2 - 4 \Rightarrow y_{2,3} = \pm\sqrt{20}/3$

Candidates for extrema: $\quad P_3 = \frac{1}{3}\begin{pmatrix} -1 \\ \sqrt{20} \end{pmatrix}\,,\ P_4 = \frac{1}{3}\begin{pmatrix} -1 \\ -\sqrt{20} \end{pmatrix}$

Since the set $g(x,y) = 0$ describes a circle, it is compact.

The continuous function f attains its absolute maximum and minimum on $g(x,y) = 0$, since the circle is a compact set. So among the candidates for extrema are the absolute maximum and minimum.

Further we compute function values at the candidates for extrema

$$f(P_1) = 36\,, \quad f(P_2) = 4\,, \quad f(P_{3,4}) = \frac{24}{9}\,.$$

So P_1 is absolute maximum and $P_{3,4}$ are absolute minima.

The intuition tells us that P_2 is the local maximum. This can be precisely determined using the sufficient condition of the second order, i.e. via the property of definiteness the Hessian matrix

$$\text{Hess}F(x, y) = \begin{pmatrix} 8 + 2\lambda & 0 \\ 0 & 2(1 + \lambda) \end{pmatrix}$$

computed in the kernel $\boldsymbol{J}\,\boldsymbol{g}\,(x, y)$ at point P_2. For $P_2 = (-1, 0)^T$ from the first equation in the Lagrange multipliers rule we obtain $0 = -8 - 4\lambda \Rightarrow \lambda = -2$. Hence we have

$$\boldsymbol{J}\,\boldsymbol{g}\,(-1, 0) = (-4, 0) \quad \text{and} \quad \text{Hess}F(-1, 0) = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} .$$

The kernel is spanned by $\boldsymbol{e}_2^T = (0, 1)$.

Since $\boldsymbol{e}_2^T \text{Hess}F(-1, 0)\,\boldsymbol{e}_2 = -2$, $\text{Hess}F$ is negative definite on the kernel. So P_2 is a strict local maximum.

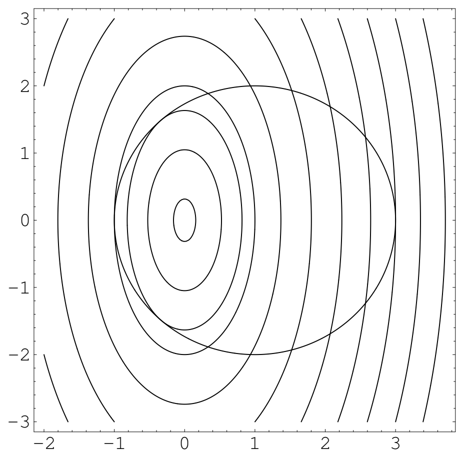

Figure 2 a) Constraint $g(x, y) := (x - 1)^2 + y^2 - 4 = 0$ with level set of function $f(x, y) = 4x^2 + y^2$

b) The circle $g(x, y) = (x - 1)^2 + y^2 - 4 = 0$ can be parameterized using the polar coordinates as follows

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\cos(t) + 1 \\ 2\sin t \end{pmatrix} =: \mathbf{c}(t) , \quad 0 \le t < 2\pi ,$$

hence it holds $g(2\cos(t) + 1, 2\sin(t)) = 0$. We search the extrema of

$$h(t) := f(\mathbf{c}(t)) = 4(2\cos(t) + 1)^2 + (2\sin t)^2 = 12\cos^2 t + 16\cos t + 8 .$$

Candidates for extrema: $h'(t) = -8\sin t(3\cos t + 2) = 0 \quad \Rightarrow$

$t_1 = 0$, $t_2 = \pi$, $t_3 = \arccos(-2/3)$, $t_4 = 2\pi - \arccos(-2/3)$.

(For $i = 1, 2, 3, 4$ the values t_i correspond to the points P_i from a).)

$h''(t) = -8(6\cos^2 t + 2\cos t - 3) \Rightarrow h''(t_1) = -40, h''(t_2) = -8, h''(t_{3,4}) = 40/3$

Hence for $t_{1,2}$ we have the local minima with the function values $h(t_1) = 36$ and $h(t_2) = 4$, and for $t_{3,4}$ local minima with the corresponding function values $h(t_{3,4}) = 24/9$.

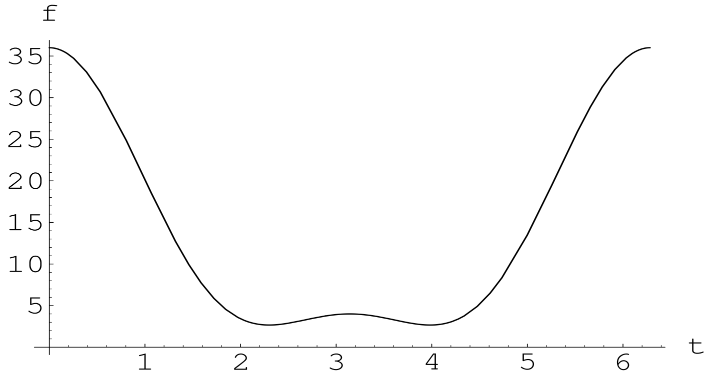

Figure 2 b) $h(t) := f(\mathbf{c}(t)) = 12\cos^2 t + 16\cos t + 8$

Discussion: 18.12. - 22.12.2023