Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Homework sheet 5, Solutions

Exercise 1:

Let $h : \mathbb{R}^3 \to \mathbb{R}$ be a function with

$$h(x, y, z) = 16z^2 + x^2 + 4y^2 + 2x - 8y + 5 \, .$$

a) Check whether the level set $g(x, y, z) = c$, defined by the point $(3, 1, 0)$ forms a smooth surface in the vicinity of this point.

b) Determine the tangent plane at the point $(3, 1, 0)$ with respect to the surface from a) in parameterized form.

c) If possible, solve the above equation for one of the variables in order to determine the area explicitly.

d) Make a sketch of the surface.

Solution:

a) Since $h(3, 1, 0) = 16$, the level set is given by the implicit equation

$$g(x, y, z) := 16z^2 + x^2 + 4y^2 + 2x - 8y - 11 = 0.$$

In order to determine whether $g(x, y, z) = 0$ forms a smooth surface in the neighbourhood of the point $(3, 1, 0)$, the assumption of the implicit function theorem must be checked:

$$\text{grad } g(x, y, z) = (2x + 2, 8y - 8, 32z)^T \quad \Rightarrow \quad \text{grad } g(3, 1, 0) = (8, 0, 0)^T \, ,$$

so only $g_x(3, 1, 0) = 8$ is an invertible 1×1 submatrix. Using the implicit function theorem, the level set forms a smooth surface that can be described by solving $g(x, y, z) = 0$ for x, i.e. in a neighborhood of $(3, 1, 0)$

$$x = f(y, z) \, , \quad \text{with} \quad f(1, 0) = 3 \quad \text{and} \quad g(f(y, z), y, z) = 0 \, .$$

b) At $(3,1,0)$ the surface f is approximately described by the associated tangent plane T_1, which in vector form is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f(y,z) \\ y \\ z \end{pmatrix} \approx \begin{pmatrix} T_1(y,z;1,0) \\ y \\ z \end{pmatrix}$$

To represent the tangent plane we need:

$$\begin{aligned} \boldsymbol{J}f(y,z) &= -(g_x)^{-1}(g_y, g_z) \\ &= -\frac{1}{2f(y,z)+2}(8y-8, 32z) \\ \Rightarrow \quad \boldsymbol{J}f(1,0) &= -\frac{1}{2\cdot 3+2}(0,0) = (0,0)\,. \end{aligned}$$

Hence the parametric form of the tangent plane is

$$\begin{aligned} \begin{pmatrix} T_1(y,z;1,0) \\ y \\ z \end{pmatrix} &= \begin{pmatrix} f(1,0) + \boldsymbol{J}f(1,0)\begin{pmatrix} y-1 \\ z \end{pmatrix} \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + (y-1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

c) $16z^2+x^2+4y^2+2x-8y+5=16$ can be explicitly solved for x. Considering $x(1,0)=3$ we can only have '+' when taking the square root.

$$\begin{aligned} 16 &= 16z^2+x^2+4y^2+2x-8y+5 \\ &= 16z^2+(x+1)^2+4(y-1)^2 \\ \Rightarrow \quad x(y,z) &= -1+\sqrt{16-16z^2-4(y-1)^2} =: h(y,z) \end{aligned}$$

There is no unique solution w.r.t y or z, because of the square root sign ambiguity.

d) $16 = 16z^2+x^2+4y^2+2x-8y+5 = 16z^2+(x+1)^2+4(y-1)^2$ Using spherical coordinates, the entire level set can be parameterized using $(\varphi,\theta) \in [0,2\pi]\times[-\pi/2,\pi/2]$ as follows

$$p(\varphi,\theta) = \begin{pmatrix} 4\cos\varphi\cos\theta - 1 \\ 2\sin\varphi\cos\theta + 1 \\ \sin\theta \end{pmatrix}$$

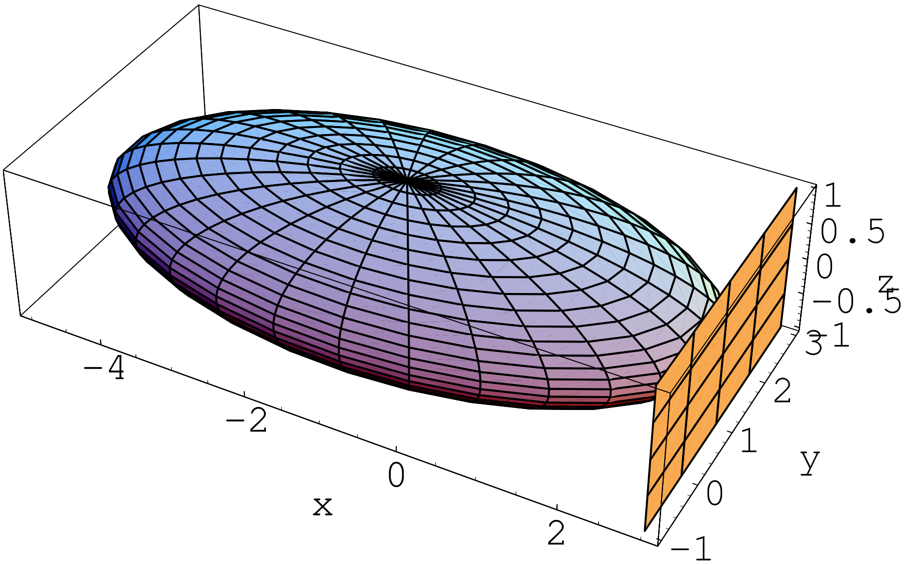

Figure 1 Level set $16z^2 + x^2 + 4y^2 + 2x - 8y + 5 = 16$ and tangent plane

Exercise 2:

For the function $f(x, y, z) = y + 2z$ compute and classify the extrema on the intersection of the parabolic cylinder $z = x^2 - 1$ with the plane $z = 2y$ using the Lagrange multipliers method.

Solution:

Constraint: $g_1(x, y, z) := x^2 - z - 1 = 0$ and $g_2(x, y, z) := z - 2y = 0$.

Regularity condition: $\boldsymbol{J}\,\boldsymbol{g}\,(x, y, z) = \begin{pmatrix} 2x & 0 & -1 \\ 0 & -2 & 1 \end{pmatrix}$

has rank 2 in all of $\mathbb{R}^3$.

All admissible points therefore satisfy the regularity condition and the Lagrange multipliers method can be applied:

Lagrange-Function: $F(x, y, z) = y + 2z + \lambda_1(x^2 - z - 1) + \lambda_2(z - 2y)$

Lagrange multipliers method:

$$\begin{pmatrix} \nabla F(x, y, z) \\ g(x, y, z) \end{pmatrix} = \begin{pmatrix} 2\lambda_1 x \\ 1 - 2\lambda_2 \\ 2 - \lambda_1 + \lambda_2 \\ x^2 - z - 1 \\ z - 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

λ_1 and λ_2 are obtained from the 2. and 3. equation by solving a system of linear equations

$$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \lambda_1 = 5/2\,,\ \lambda_2 = 1/2$$

1. Equation: $x = 0 \Rightarrow 0 = g_1(0, y, z) = -z - 1 \Rightarrow z = -1$
$\Rightarrow 0 = g_2(0, y, -1) = -1 - 2y \Rightarrow y = -1/2$

The only candidate for extrema: $P_1 = \begin{pmatrix} 0 \\ -1/2 \\ -1 \end{pmatrix}$.

Since the intersection of the parabolic cylinder $z = x^2 - 1$ with the plane $z = 2y$ is a parabola, which is not bounded, one can not use here the compactness argument.

We classify the candidate for extrema P_1 using the sufficient second-order condition, i.e. via the definiteness property of the Hessian matrix

$$\text{Hess}F(x, y, z) = \begin{pmatrix} 2\lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with the kernel $\boldsymbol{J}\,\boldsymbol{g}\,(x, y, z)$ at P_1

$$\boldsymbol{J}\,\boldsymbol{g}\,(0, -1/2, -1) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 1 \end{pmatrix}.$$

The kernel is spanned by $\boldsymbol{e}_1^T = (1, 0, 0)$.

Since $\boldsymbol{e}_1^T \text{Hess} F(0, -1/2, -1)\, \boldsymbol{e}_1 = 2\lambda_1 = 5$, the $\text{Hess} F$ is positive definite on the tangent space. So P_1 is a strict local minimum with the function value $f(P_1) = -5/2$.

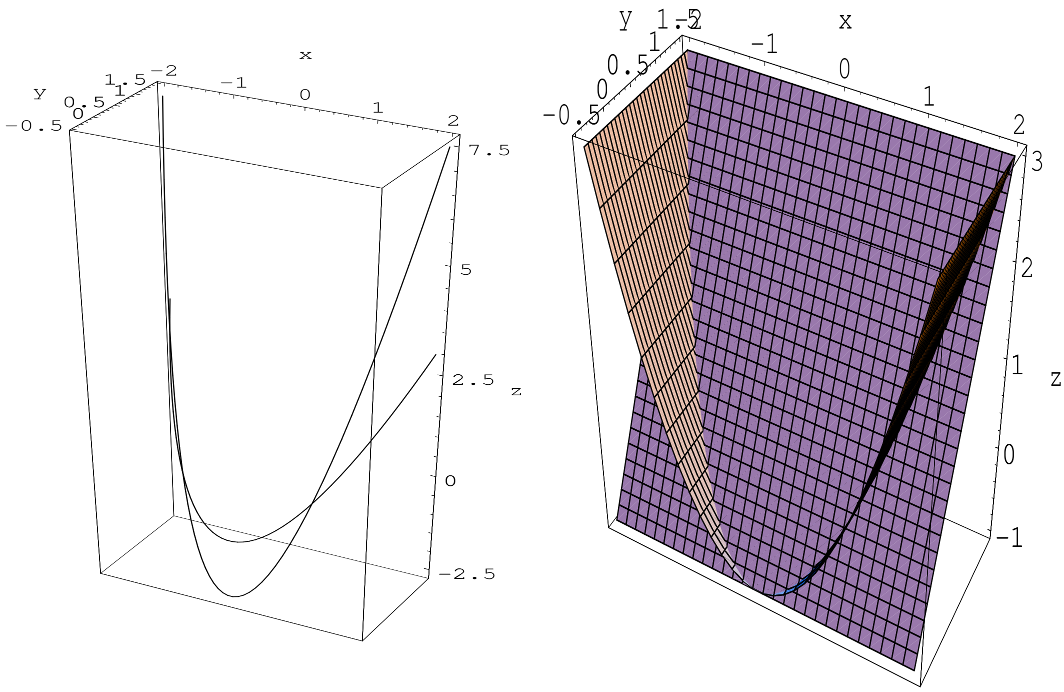

Figure 2: f on the intersection of the parabolic cylinder $z = x^2 - 1$ with the plane $z = 2y$

Submission deadline: 22.12.2023