Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Work Sheet 4, Solutions

Exercise 1:

Compute the Taylor polynomial of second degree for the function

$$f(x, y) = (y + \cos y)\sin x$$

about the point $(x_0, y_0) = \left(\frac{\pi}{2}, 0\right)$ and provide the upper bound for the error of approximation of f by T_2 at point $(x, y) = (0, 0)$.

Solution:

$f(x, y) = (y + \cos y)\sin x \quad \Rightarrow \quad f\left(\frac{\pi}{2}, 0\right) = 1$

$f_x(x, y) = (y + \cos y)\cos x \quad \Rightarrow \quad f_x\left(\frac{\pi}{2}, 0\right) = 0$

$f_y(x, y) = (1 - \sin y)\sin x \quad \Rightarrow \quad f_y\left(\frac{\pi}{2}, 0\right) = 1$

$f_{xx}(x, y) = -(y + \cos y)\sin x \quad \Rightarrow \quad f_{xx}\left(\frac{\pi}{2}, 0\right) = -1$

$f_{xy}(x, y) = (1 - \sin y)\cos x \quad \Rightarrow \quad f_{xy}\left(\frac{\pi}{2}, 0\right) = 0$

$f_{yy}(x, y) = -\cos y \sin x \quad \Rightarrow \quad f_{yy}\left(\frac{\pi}{2}, 0\right) = -1$

$$
\begin{aligned}
T_2\left(x,y;\frac{\pi}{2},0\right) &= f\left(\frac{\pi}{2},0\right) + f_x\left(\frac{\pi}{2},0\right)\left(x-\frac{\pi}{2}\right) + f_y\left(\frac{\pi}{2},0\right) y \\
&\quad + \frac{1}{2}\left(f_{xx}\left(\frac{\pi}{2},0\right)\left(x-\frac{\pi}{2}\right)^2 + 2f_{xy}\left(\frac{\pi}{2},0\right)\left(x-\frac{\pi}{2}\right) y + f_{yy}\left(\frac{\pi}{2},0\right) y^2\right) \\
&= 1 + y - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2 - \frac{y^2}{2}
\end{aligned}
$$

The third derivatives are required for the error estimation

$$
\begin{aligned}
f_{xxx}(x,y) &= -(y+\cos y)\cos x \\
f_{xxy}(x,y) &= -(1-\sin y)\sin x \\
f_{xyy}(x,y) &= -\cos y\cos x \\
f_{yyy}(x,y) &= \sin y \sin x.
\end{aligned}
$$

We estimate the error at $(x,y)=(0,0)$ with $\theta \in]0,1[$, $(\xi_1,\xi_2) := \left((1-\theta)\frac{\pi}{2},0\right)$. So it holds $0<\xi_1<\frac{\pi}{2}$ and $\xi_2 = 0$.

$$
\begin{aligned}
&\left|f(0,0) - T_2\left(0,0;\frac{\pi}{2},0\right)\right| = \left|R_2\left(0,0;\frac{\pi}{2},0\right)\right| \\
&= \frac{1}{3!}\left|f_{xxx}(\xi_1,0)\left(0-\frac{\pi}{2}\right)^3 + 3f_{xxy}(\xi_1,0)\left(0-\frac{\pi}{2}\right)^2\cdot(0-0)\right. \\
&\quad \left. +3f_{xyy}(\xi_1,0)\left(0-\frac{\pi}{2}\right)\cdot(0-0)^2 + f_{yyy}(\xi_1,0)(0-0)^3\right| \\
&= \frac{1}{3!}\left|f_{xxx}(\xi_1,0)\right|\cdot\left|-\frac{\pi}{2}\right|^3 = \frac{\pi^3|-(0+\cos 0)\cos\xi_1|}{48} \le \frac{\pi^3}{48} = 0.645964...
\end{aligned}
$$

The actual error is

$$
\left|f(0,0) - T_2\left(0,0;\frac{\pi}{2},0\right)\right| = \left|T_2\left(0,0;\frac{\pi}{2},0\right)\right| = \left|1-\frac{1}{2}\left(-\frac{\pi}{2}\right)^2\right| = 0.233701...
$$

Exercise 2:

Compute and classify all stationary points of the following functions

a) $f(x,y) = \dfrac{3x^2}{2} + x^3 - y^3 + 3y\,,$

b) $f(x,y) = 2(x^2+y^2)^2 - x^2 - y^2\,,$

c) $f(x,y) = \sqrt{x^2+y^2}\,,$

d) $f(x,y) = x\sin y\,.$

Solution:

a) grad $f(x,y) = (3x+3x^2, -3y^2+3)^T = (3x(1+x), 3(1-y^2))^T = (0,0)^T$

We obtain $x=0$ or $x=-1$ and $y=1$ or $y=-1$, so the points

$$P_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ P_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ P_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ P_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Hess $f(x,y) = \begin{pmatrix} 3+6x & 0 \\ 0 & -6y \end{pmatrix}$

Hess $f(P_1) = \begin{pmatrix} 3 & 0 \\ 0 & -6 \end{pmatrix} \Rightarrow$ indefinite $\Rightarrow P_1$ saddle point

Hess $f(P_2) = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \Rightarrow$ positive definite $\Rightarrow P_2$ minimum

Hess $f(P_3) = \begin{pmatrix} -3 & 0 \\ 0 & -6 \end{pmatrix} \Rightarrow$ negative definite $\Rightarrow P_3$ maximum

Hess $f(P_4) = \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix} \Rightarrow$ indefinite $\Rightarrow P_4$ saddle point

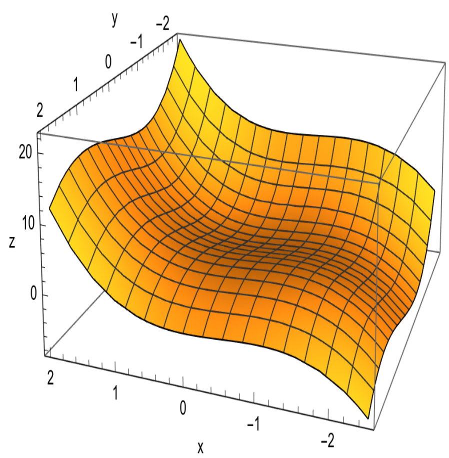

Figure 2 a): $f(x,y) = \dfrac{3x^2}{2} + x^3 - y^3 + 3y$

b) $f(x,y) = 2(x^2+y^2)^2 - x^2 - y^2 \Rightarrow \text{grad } f(x,y) = (4(x^2+y^2)-1)(2x,2y)^T = (0,0)$

$\Rightarrow$ stationary points are $(x_0, y_0) = (0,0)$ and all points with

$$4(x^2+y^2) - 1 = 0 \Leftrightarrow x^2+y^2 = \left(\frac{1}{2}\right)^2 .$$

So they are on the circle K of radius $r = \frac{1}{2}$ with center at $(0,0)$.

$$\boldsymbol{H}\, f(x,y) = \begin{pmatrix} 2(4(x^2+y^2)-1)+16x^2 & 16xy \\ 16xy & 2(4(x^2+y^2)-1)+16y^2 \end{pmatrix}$$

$\boldsymbol{H}\, f(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ is negative definite

$\Rightarrow \quad (x_0, y_0) = (0,0)$ is strict local maximum.

For the stationary points at K we have with $x^2+y^2 = \frac{1}{4}$:

$$\boldsymbol{H}\, f(x,y) = \begin{pmatrix} 16x^2 & 16xy \\ 16xy & 16y^2 \end{pmatrix} .$$

We obtain the eigenvalues from the characteristic polynomial

$$p(\lambda) = (16x^2 - \lambda)(16y^2 - \lambda) - 16^2x^2y^2 = \lambda^2 - 16(x^2+y^2)\lambda = \lambda(\lambda - 4) = 0$$

$\Rightarrow \quad \lambda_1 = 0\,, \lambda_2 = 4 \quad \Rightarrow \quad \boldsymbol{H}\, f(x,y)$ ist positive semidefinite.

Hence the points from K can not be local maxima.

Since $r^2 = x^2+y^2$ and $\tilde{f}(r) = 2r^4 - r^2$ we know that f is rotationally symmetric. $\tilde{f}$ has absolute minima for $r = \pm\frac{1}{2}$ and for $r = 0$ a strict local maximum.

So the points on K are absolute minima (not strict).

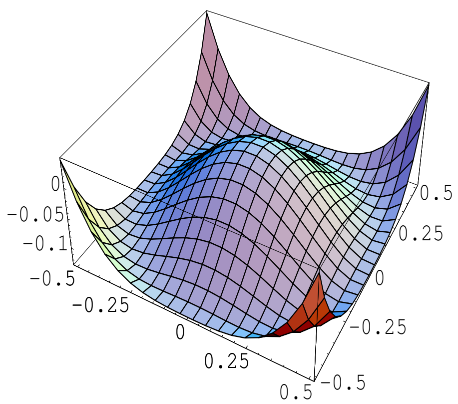

Figure 2 b): $f(x,y) = 2(x^2+y^2)^2 - x^2 - y^2$

c) $f(x,y) = \sqrt{x^2+y^2}$

$$\text{grad } f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)^T = (0,0)^T \quad \Rightarrow \quad (x_0,y_0) = (0,0)$$

is not admissible, since grad f in $(0,0)$ does not exist.

So there is no stationary points.

The only remaining point $(0,0)$ is obviously strict global minimum, since $f(0,0) = 0$ and $f(x,y) > 0 \Leftrightarrow (x,y) \neq (0,0)$.

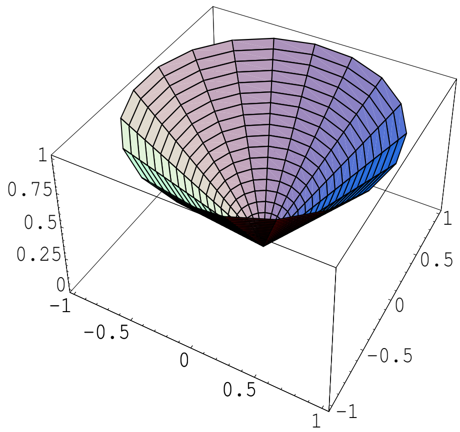

Figure 2 c): $f(x,y) = \sqrt{x^2+y^2}$

d) $f(x,y) = x\sin y$.

grad $f(x,y) = (\sin y, x\cos y)^T = (0,0)^T$

$$\Rightarrow \quad \sin y = 0 \quad \Rightarrow \quad y_k = k\pi \quad \Rightarrow \quad \cos(k\pi) = (-1)^k \quad \Rightarrow \quad x_k = 0$$

So the stationary points are given by $(x_k, y_k) = (0, k\pi)$

$$\boldsymbol{H}\, f(x,y) = \begin{pmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{pmatrix}$$

$$\Rightarrow \boldsymbol{H}\, f(0,k\pi) = \begin{pmatrix} 0 & (-1)^k \\ (-1)^k & 0 \end{pmatrix}$$

has eigenvalues $\lambda_{1,2} = \pm 1$.

Hence $\boldsymbol{H}\, f(0,k\pi)$ is indefinite, all stationary points are saddle points.

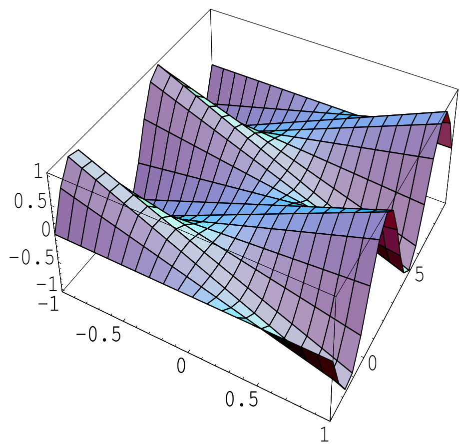

Figure 2 d): $f(x,y) = x\sin y$

Discussion: 4.12. - 8.12.2023