Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Homework Sheet 4, Solutions

Exercise 1:

a) Compute Taylor polynomial of second degree about a point $(x_0, y_0, z_0) = (0,0,0)$ of the following function
$$f(x,y,z) = x - y + (x-z)^2 + (y-z)^3 \,.$$

b) Compute Taylor polynomial of third degree of the following function
$$f(x,y) = x + (y+1)\cosh(x+y)$$
about a point $(0,0)$.

Solution:

a)
$$\begin{array}{lcl}
f(x,y,z) = x - y + (x-z)^2 + (y-z)^3 & \Rightarrow & f(0,0,0) = 0 \\[1em]
f_x(x,y,z) = 1 + 2(x-z) & \Rightarrow & f_x(0,0,0) = 1 \\
f_y(x,y,z) = -1 + 3(y-z)^2 & \Rightarrow & f_y(0,0,0) = -1 \\
f_z(x,y,z) = -2(x-z) - 3(y-z)^2 & \Rightarrow & f_y(0,0,0) = 0 \\[1em]
f_{xx}(x,y,z) = 2 & \Rightarrow & f_{xx}(0,0,0) = 2 \\
f_{xy}(x,y,z) = 0 & \Rightarrow & f_{xy}(0,0,0) = 0 \\
f_{xz}(x,y,z) = -2 & \Rightarrow & f_{xz}(0,0,0) = -2 \\
f_{yy}(x,y,z) = 6(y-z) & \Rightarrow & f_{yy}(0,0,0) = 0 \\
f_{yz}(x,y,z) = -6(y-z) & \Rightarrow & f_{yz}(0,0,0) = 0 \\
f_{zz}(x,y,z) = 2 + 6(y-z) & \Rightarrow & f_{zz}(0,0,0) = 2
\end{array}$$

$$\begin{aligned}
\Rightarrow T_2(x,y,z;0,0,0) \;&=\; f(0,0,0) + f_x(0,0,0)x + f_y(0,0,0)y + f_z(0,0,0)z \\
&\quad + \frac{1}{2}\big(f_{xx}(0,0,0)x^2 + f_{yy}(0,0,0)y^2 + f_{zz}(0,0,0)z^2 \\
&\quad + 2f_{xy}(0,0,0)xy + 2f_{xz}(0,0,0)xz + 2f_{yz}(0,0,0)yz\big) \\[1em]
&=\; x - y + x^2 - 2xz + z^2
\end{aligned}$$

Since we are expanding the function about the origin opening brackets leads to some terms cancelling out

$$f(x, y, z) = x - y + (x - z)^2 + (y - z)^3 = x - y + x^2 - 2xz + z^2 + (y - z)^3$$

b)

$$
\begin{aligned}
f(x, y) &= x + (y + 1)\cosh(x + y) &&\Rightarrow\ f\,(0, 0) = 1 \\
f_x(x, y) &= 1 + (1 + y)\sinh(x + y) &&\Rightarrow\ f_x\,(0, 0) = 1 \\
f_y(x, y) &= \cosh(x + y) + (1 + y)\sinh(x + y) &&\Rightarrow\ f_y\,(0, 0) = 1
\end{aligned}
$$

$$
\begin{aligned}
f_{xx}(x, y) &= (1 + y)\cosh(x + y) &&\Rightarrow\ f_{xx}\,(0, 0) = 1 \\
f_{xy}(x, y) &= (1 + y)\cosh(x + y) + \sinh(x + y) &&\Rightarrow\ f_{xy}\,(0, 0) = 1 \\
f_{yy}(x, y) &= (1 + y)\cosh(x + y) + 2\sinh(x + y) &&\Rightarrow\ f_{yy}\,(0, 0) = 1
\end{aligned}
$$

$$
\begin{aligned}
f_{xxx}(x, y) &= (1 + y)\sinh(x + y) &&\Rightarrow\ f_{xxx}\,(0, 0) = 0 \\
f_{xxy}(x, y) &= \cosh(x + y) + (1 + y)\sinh(x + y) &&\Rightarrow\ f_{xxy}\,(0, 0) = 1 \\
f_{xyy}(x, y) &= 2\cosh(x + y) + (1 + y)\sinh(x + y) &&\Rightarrow\ f_{xyy}\,(0, 0) = 2 \\
f_{yyy}(x, y) &= 3\cosh(x + y) + (1 + y)\sinh(x + y) &&\Rightarrow\ f_{yyy}\,(0, 0) = 3
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow T_3(x, y; 0, 0) &= f(0, 0) + f_x(0, 0)x + f_y(0, 0)y \\
&\quad + \frac{1}{2}\left(f_{xx}(0, 0)x^2 + 2f_{xy}(0, 0)xy + f_{yy}(0, 0)y^2\right) \\
&\quad + \frac{1}{6}\left(f_{xxx}(0, 0)x^3 + 3f_{xxy}(0, 0)x^2y\right. \\
&\quad \left. + 3f_{xyy}(0, 0)xy^2 + f_{yyy}(0, 0)y^3\right) \\
&= 1 + x + y + x^2/2 + xy + y^2/2 + x^2y/2 + xy^2 + y^3/2
\end{aligned}
$$

Exercise 2:

Given the function $f(x, y) = 9x^4 - 12x^2y + 4y^2$

a) compute all stationary points of f,

b) try to classify stationary points using the sufficient optimality condition,

c) prove that f has a strict local minimum along every straight line going through zero,

d) classify all stationary points of f,

e) plot the function for example using Matlab commands 'ezsurf' and 'ezcontour'.

Solution:

a) $\operatorname{grad} f(x, y) = (3x^2 - 2y)(12x, -4)^T = (0, 0)^T$.
On the parabola $3x^2 - 2y = 0$ all points are stationary.

b) $\boldsymbol{H} f(x, y) = \begin{pmatrix} 12(9x^2 - 2y) & -24x \\ -24x & 8 \end{pmatrix}$

$\Rightarrow \quad \boldsymbol{H} f(x, 3x^2/2) = \begin{pmatrix} 72x^2 & -24x \\ -24x & 8 \end{pmatrix}$

Since $p(\lambda) = (72x^2 - \lambda)(8 - \lambda) - 24^2x^2 = \lambda(\lambda - (8 + 72x^2))$ the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 8 + 72x^2$.

So the Hessian is positive semidefinite and the sufficient condition is not applicable. The necessary condition still allows for the stationary points on the parabola $3x^2 - 2y = 0$ to be minimum or saddle points.

c) On the straight lines $x = 0$ and $y = 0$ the function f is given by

$$g(y) := f(0, y) = 4y^2 \quad \text{and} \quad \tilde{g}(x) := f(x, 0) = 9x^4.$$

For $y = 0$ and $x = 0$ these functions have a strict local minimum.

All other lines going through the origin can be represented as $y = ax$ with $a \neq 0$ and the function then can be given as

$$h(x) := f(x, ax) = 9x^4 - 12ax^3 + 4a^2x^2.$$

We obtain

$$h'(x) = 36x^3 - 36ax^2 + 8a^2x \quad \Rightarrow \quad h'(0) = 0$$

and

$$h''(x) = 108x^2 - 72ax + 8a^2 \quad \Rightarrow \quad h''(0) = 8a^2 > 0\,.$$

Hence h has at $x = 0$ a strict local minimum.

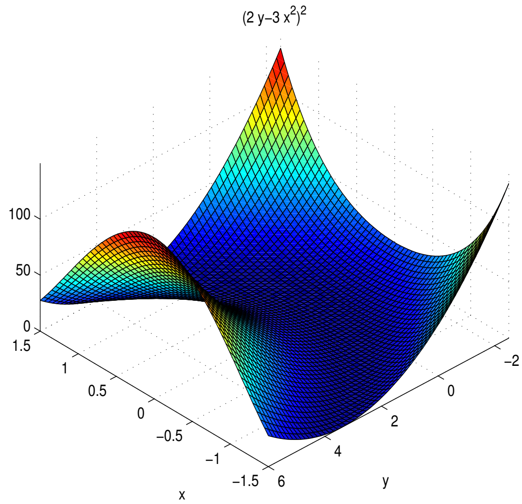
However, this does not lead to the conclusion that f has a strict local minimum at $(0, 0)$.

d) Since

$$f(x,y) = 9x^4 - 12x^2y + 4y^2 = (3x^2 - 2y)^2 \geq 0$$

it holds $f(x, 3x^2/2) = 0$. So all points on the parabola $3x^2 - 2y = 0$ are global but not strict minima.

e)

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ezsurf('9*x^4 -12*x^2*y +4*y^2',[-1.5,1.5,-2.5,6])
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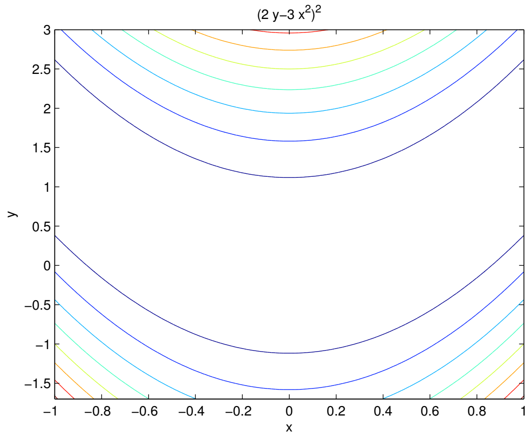
Figure 2 a) $f(x,y) = 9x^4 - 12x^2y + 4y^2$




```
ezcontour('9*x^4 -12*x^2*y +4*y^2',[-1,1,-1.6,3])
```

Figure 2 b) $f(x, y) = 9x^4 - 12x^2y + 4y^2$

Submission deadline: 8.12.2023