**Department of Mathematics, University of Hamburg** **WiSe 2023/24**
Prof. Dr. J. Struckmeier
Dr. K. Rothe, Md. T. Hassan

# Analysis III for Engineering Students

## Work sheet 3, Solutions

**Exercise 1:**

a) Compute the Jacobian matrix of $h$ using the chain rule:

$$h:\ \mathbb{R}^2 \overset{\boldsymbol{f}}{\to} \mathbb{R}^2 \overset{g}{\to} \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \mapsto g(u,v)\,.$$

b) Compute the Jacobian matrix of $f$ directly and also using the chain rule

$$\boldsymbol{f}:\ \mathbb{R}^2 \overset{\boldsymbol{f}_1}{\to} \mathbb{R}^3 \overset{\boldsymbol{f}_2}{\to} \mathbb{R}^2$$

$$\begin{pmatrix} r \\ s \end{pmatrix} \mapsto \begin{pmatrix} u = \sin(rs) \\ v = e^r + s \\ w = 1 - 2s^3 \end{pmatrix} \mapsto \begin{pmatrix} uw \\ vw \end{pmatrix}.$$

**Solution:**

a) Chain rule: $h = g \circ \boldsymbol{f}$ i.e. $h(x,y) = g(u(x,y), v(x,y))$

$$\boldsymbol{J}\ \boldsymbol{f}\,(x,y) = \begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix},$$

$$\boldsymbol{J}\, g(u,v) = (g_u(u,v), g_v(u,v))$$

$$\Rightarrow \quad \boldsymbol{J}h(x,y) = (h_x(x,y), h_y(x,y)) = \boldsymbol{J}g(u,v)\cdot\boldsymbol{J}\boldsymbol{f}(x,y)$$

$$= (g_u(u,v), g_v(u,v))\begin{pmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{pmatrix}$$

$$= (g_u(u,v)\cdot u_x(x,y) + g_v(u,v)\cdot v_x(x,y),$$

$$g_u(u,v)\cdot u_y(x,y) + g_v(u,v)\cdot v_y(x,y))$$

$$= (g_u u_x + g_v v_x\ ,\ g_u u_y + g_v v_y) = (h_x\ ,\ h_y)$$

b) Chain rule: $\boldsymbol{f} = \boldsymbol{f}_2 \circ \boldsymbol{f}_1$

$$\boldsymbol{J}\ \boldsymbol{f}_1(r,s) = \begin{pmatrix} s\cos(rs) & r\cos(rs) \\ e^r & 1 \\ 0 & -6s^2 \end{pmatrix}\ ,\ \boldsymbol{J}\ \boldsymbol{f}_2(u,v,w) = \begin{pmatrix} w & 0 & u \\ 0 & w & v \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{J}\boldsymbol{f}(r,s) &= \boldsymbol{J}(\boldsymbol{f}_2\circ\boldsymbol{f}_1)(r,s) = \boldsymbol{J}\boldsymbol{f}_2(\boldsymbol{f}_1(r,s))\cdot\boldsymbol{J}\boldsymbol{f}_1(r,s) \\
&= \begin{pmatrix} w & 0 & u \\ 0 & w & v \end{pmatrix}\begin{pmatrix} s\cos(rs) & r\cos(rs) \\ e^r & 1 \\ 0 & -6s^2 \end{pmatrix} \\
&= \begin{pmatrix} ws\cos(rs) & wr\cos(rs) - 6us^2 \\ we^r & w - 6vs^2 \end{pmatrix} \\
&= \begin{pmatrix} (1-2s^3)s\cos(rs) & (1-2s^3)r\cos(rs) - 6s^2\sin(rs) \\ e^r(1-2s^3) & 1-2s^3-6(e^r+s)s^2 \end{pmatrix}
\end{aligned}$$

directly:

$$\begin{aligned}
\boldsymbol{f}_2(\boldsymbol{f}_1(r,s)) &= \boldsymbol{f}_2(u(r,s), v(r,s), w(r,s)) \\
&= \boldsymbol{f}(r,s) = \begin{pmatrix} \sin(rs)(1-2s^3) \\ (e^r+s)(1-2s^3) \end{pmatrix}
\end{aligned}$$

$$\Rightarrow \quad \boldsymbol{J}\ \boldsymbol{f}\,(r,s) = \begin{pmatrix} s\cos(rs)(1-2s^3) & r\cos(rs)(1-2s^3) - 6s^2\sin(rs) \\ e^r(1-2s^3) & (1-2s^3) - 6s^2(e^r+s) \end{pmatrix}$$

**Exercise 2:**

a) Draw the following circles and ellipses

(i) $x^2 + y^2 = 5$,

(ii) $16x^2 + 25y^2 = 400$,

(iii) $x^2 + y^2 + 6(x - y) + 9 = 0$,

(iv) $x^2 + 2y^2 - 16y + 28 = 0$.

Determine the $(x, y)$ of the solution sets of the above equations using polar coordinates.

b) Draw the solution sets of the following areas in $\mathbb{R}^3$

(i) $x \leq 0,\ y \leq 0,\ 0 \leq z$ and $x^2 + y^2 + z^2 \leq 9$

(ii) $1 \leq z \leq 2,\ 0 \leq y$ and $x^2 + y^2 \leq 9$

and represent them using cylindrical or spherical coordinates.

**Solution:**

a) (i) $x^2 + y^2 = 5$

describes the circle

with radius $r = \sqrt{5}$

and a center at $(0, 0)$

$(x, y) = (\sqrt{5}\cos(\varphi), \sqrt{5}\sin(\varphi))$

with $-\pi \leq \varphi < \pi$

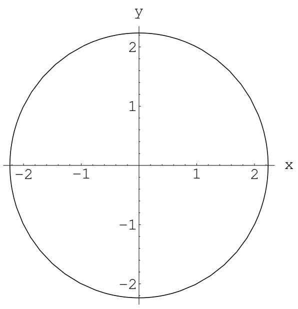


**Figure 2.1** Circle $x^2 + y^2 = 5$

(ii) $16x^2 + 25y^2 = 400$

$\Leftrightarrow \quad \dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$

describes the ellipse

with the semi-axes

$a = 5$ and $b = 4$

with center at $(0, 0)$.

$(x, y) = (5\cos(\varphi), 4\sin(\varphi))$

with $-\pi \leq \varphi < \pi$

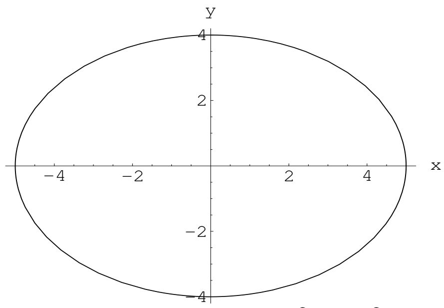


**Figure 2.2** Ellipse $\dfrac{x^2}{5^2} + \dfrac{y^2}{4^2} = 1$

(iii) By completing the square we obtain

$$0 = x^2 + y^2 + 6(x-y) + 9$$
$$= x^2 + 6x + 9 + y^2 - 6y + 9 - 9$$
$$\Leftrightarrow \ (x+3)^2 + (y-3)^2 = 9\,.$$

So this is a circle

with a center at $(-3, 3)$

and radius $r = 3$.

$(x, y) = (3\cos(\varphi) - 3, 3\sin(\varphi) + 3)$

with $-\pi \le \varphi < \pi$

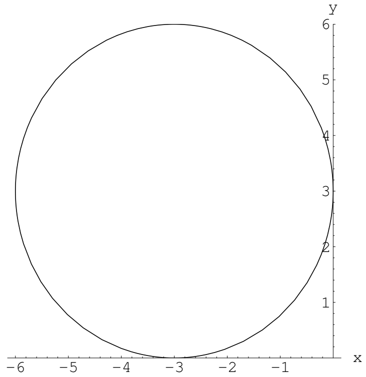


**Figure 2.3** Circle $(x+3)^2 + (y-3)^2 = 9$

(iv) By completing the square we obtain

$$x^2 + 2y^2 - 16y + 28$$
$$= x^2 + 2(y^2 - 8y + 16) - 4 = 0$$
$$\Leftrightarrow \ \left(\frac{x}{2}\right)^2 + \left(\frac{y-4}{\sqrt{2}}\right)^2 = 1.$$

This is an ellipse

with the center at $(0, 4)$

with the semi-axes

$a = 2$ and $b = \sqrt{2}$

$(x, y) = (2\cos(\varphi), \sqrt{2}\sin(\varphi) + 4)$

with $-\pi \le \varphi < \pi$

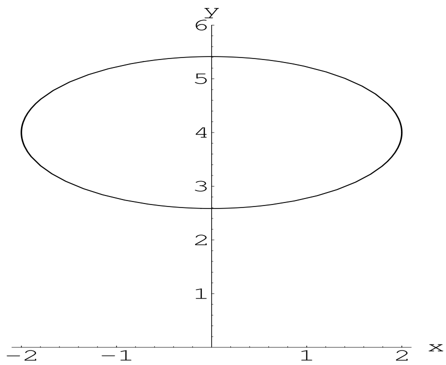


**Figure 2.4** Ellipse $\left(\frac{x}{2}\right)^2 + \left(\frac{y-4}{\sqrt{2}}\right)^2 = 1$

b) (i) $x \le 0,\ y \le 0,\ 0 \le z$ and $x^2 + y^2 + z^2 \le 9$

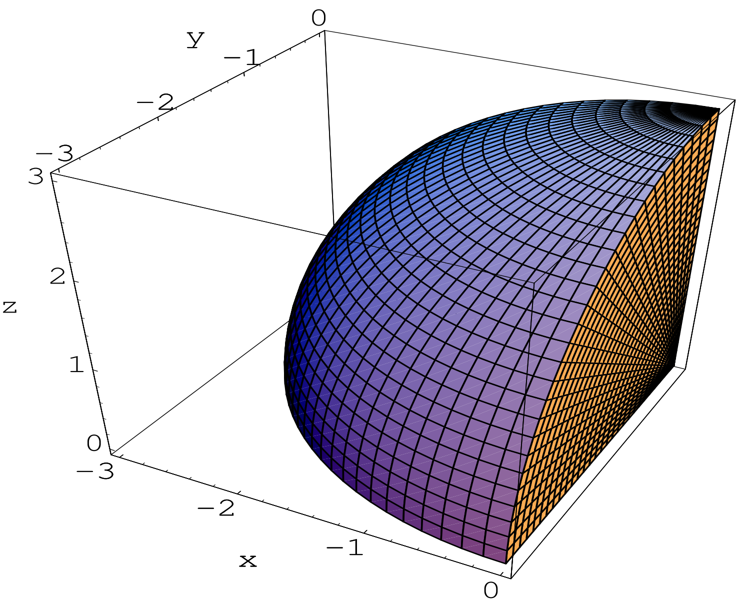


**Figure 2.5** Eighth of a ball $K$

Spherical coordinates for $K$: $\quad \boldsymbol{u} = (r, \varphi, \theta)^T$

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{\Phi}(r, \varphi, \theta) = \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ r\sin(\varphi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$$

with $\quad 0 \le r \le 3\,, \quad -\pi \le \varphi \le -\dfrac{\pi}{2}\,, \quad 0 \le \theta \le \dfrac{\pi}{2}$

(ii) $1 \le z \le 2$, $0 \le y$ and $x^2 + y^2 \le 9$

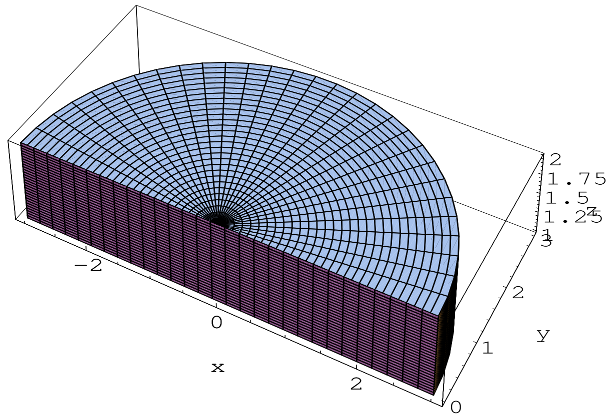


**Figure 2.6** Half cylinder $Z$

Cylindrical coordinates for $Z$: $\quad \boldsymbol{u} = (r, \varphi, z)^T$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix} = \boldsymbol{\Phi}(r, \varphi, z)$$

with $\quad 0 \le r \le 3\,, \quad 0 \le \varphi \le \pi\,, \quad 1 \le z \le 2$

**Discussion:** 20.11. - 24.11.23