Department of Mathematics, University of Hamburg **WiSe 2023/24**
Prof. Dr. J. Struckmeier
Dr. K. Rothe, Md. T. Hassan

Analysis III for Engineering Students

Homework sheet 3, Solutions

Exercise 1:

Compute the derivative in direction $\boldsymbol{h} = (h_1, h_2)^T$ for the function $f : \mathbb{R}^2 \to \mathbb{R}$ $f(x, y) = x^2 + y$ at the point (x_0, y_0). What is the slope of the function at the point $(2, -3)$ in the directions given by the straight line $2x + 7y = 3$?

Solution:

Since f is continuously partially differentiable, the directional derivative can be calculated at point (x_0, y_0) as follows:

$$D_{\boldsymbol{h}} f(x_0, y_0) = < \text{grad } f(x_0, y_0), \boldsymbol{h} >= (2x_0, 1) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 2x_0 h_1 + h_2$$

The straight line $2x + 7y = 3$ in parametric form is given by:

$$\boldsymbol{g}(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix} = \begin{pmatrix} x \\ 3/7 - 2x/7 \end{pmatrix} = \begin{pmatrix} 0 \\ 3/7 \end{pmatrix} + x \begin{pmatrix} 1 \\ -2/7 \end{pmatrix}$$

To calculate the slope, we need to normalize first the direction vector $\boldsymbol{h}$ from the straight line equation:

$$\boldsymbol{h} = \pm \frac{7}{\sqrt{53}} \begin{pmatrix} 1 \\ -2/7 \end{pmatrix} = \pm \frac{1}{\sqrt{53}} \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

The ascent or descent at the point $(x_0, y_0) = (2, -3)$ is therefore

$$D_{\boldsymbol{h}} f(2, -3) = 4h_1 + h_2 = \pm \left(\frac{28}{\sqrt{53}} - \frac{2}{\sqrt{53}} \right) = \pm \frac{26}{\sqrt{53}}\,.$$

Exercise 2:

Given the coordinate transformation

$$\mathbf{\Phi}(r,\varphi) = \begin{pmatrix} x(r,\varphi) \\ y(r,\varphi) \end{pmatrix} = \begin{pmatrix} 2r\cos\varphi \\ 3r\sin\varphi \end{pmatrix}$$

with $(r,\varphi) \in Q :=]0,1] \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

a) Compute $\boldsymbol{J}\,\mathbf{\Phi}(r,\varphi)$ and $\det(\boldsymbol{J}\,\mathbf{\Phi}(r,\varphi))$ as well as

b) $\mathbf{\Phi}^{-1}(x,y)$, $\boldsymbol{J}\,\mathbf{\Phi}^{-1}(x,y)$ and $\det(\boldsymbol{J}\,\mathbf{\Phi}^{-1}(x,y))$.

c) Make a sketch of Q and $\mathbf{\Phi}(Q)$.

Solution:

a) $\boldsymbol{J}\,\mathbf{\Phi}(r,\varphi) = \begin{pmatrix} x_r & x_\varphi \\ y_r & y_\varphi \end{pmatrix} = \begin{pmatrix} 2\cos\varphi & -2r\sin\varphi \\ 3\sin\varphi & 3r\cos\varphi \end{pmatrix}$,

$\det(\boldsymbol{J}\,\mathbf{\Phi}(r,\varphi)) = 6r$

b) So $\mathbf{\Phi}$ represents elliptic coordinates, i.e. it holds

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = r^2 \overset{0<r\le 1}{\Rightarrow} r(x,y) = \sqrt{x^2/4 + y^2/9} = \left(\sqrt{9x^2+4y^2}\right)/6$$

$$\frac{y}{x} = \frac{3r\sin\varphi}{2r\cos\varphi} = \frac{3}{2}\tan\varphi \overset{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}{\Rightarrow} \varphi(x,y) = \arctan\frac{2y}{3x}$$

$$\mathbf{\Phi}^{-1}(x,y) = \begin{pmatrix} r(x,y) \\ \varphi(x,y) \end{pmatrix} = \begin{pmatrix} \left(\sqrt{9x^2+4y^2}\right)/6 \\ \arctan\frac{2y}{3x} \end{pmatrix},$$

$$\boldsymbol{J}\,\mathbf{\Phi}^{-1}(x,y) = \begin{pmatrix} r_x & r_y \\ \varphi_x & \varphi_y \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\cdot 9x}{2\cdot 6\sqrt{9x^2+4y^2}} & \frac{2\cdot 4y}{2\cdot 6\sqrt{9x^2+4y^2}} \\ -\frac{2y}{3x^2(1+(2y/3x)^2)} & \frac{2}{3x(1+(2y/3x)^2)} \end{pmatrix} = \begin{pmatrix} \frac{3x}{2\sqrt{9x^2+4y^2}} & \frac{2y}{3\sqrt{9x^2+4y^2}} \\ -\frac{6y}{9x^2+4y^2} & \frac{6x}{9x^2+4y^2} \end{pmatrix}$$

$\det\left(\boldsymbol{J}\,\mathbf{\Phi}^{-1}(x,y)\right)$

$$= \frac{3x}{2\sqrt{9x^2+4y^2}} \cdot \frac{6x}{9x^2+4y^2} - \frac{2y}{3\sqrt{9x^2+4y^2}} \cdot \left(-\frac{6y}{9x^2+4y^2}\right)$$

$$= \frac{1}{\sqrt{9x^2+4y^2}}$$

Alternatively:

$$\boldsymbol{J}\,\mathbf{\Phi}^{-1}(x,y) = (\boldsymbol{J}\,\mathbf{\Phi}(r(x,y),\varphi(x,y)))^{-1}$$

$$= \frac{1}{6r}\begin{pmatrix} 3r\cos\varphi & 2r\sin\varphi \\ -3\sin\varphi & 2\cos\varphi \end{pmatrix}$$

$$= \frac{1}{6r}\begin{pmatrix} 3x/2 & 2y/3 \\ -y/r & x/r \end{pmatrix} = \begin{pmatrix} \frac{3x}{2\sqrt{9x^2+4y^2}} & \frac{2y}{3\sqrt{9x^2+4y^2}} \\ -\frac{6y}{9x^2+4y^2} & \frac{6x}{9x^2+4y^2} \end{pmatrix}$$

$$\det(\boldsymbol{J}\ \boldsymbol{\Phi}^{-1}(x,y)) = \frac{1}{\det(\boldsymbol{J}\ \boldsymbol{\Phi}\,(r,\varphi))} = \frac{1}{6r} = \frac{1}{\sqrt{9x^2+4y^2}}\,.$$

c)

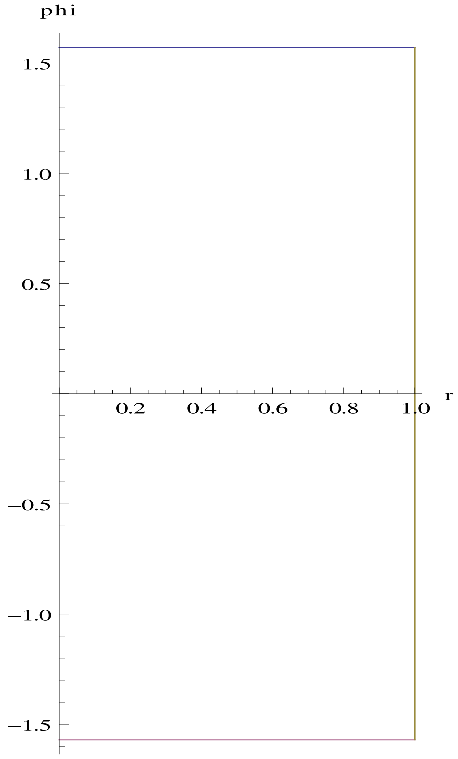

Figure 2 a: Q

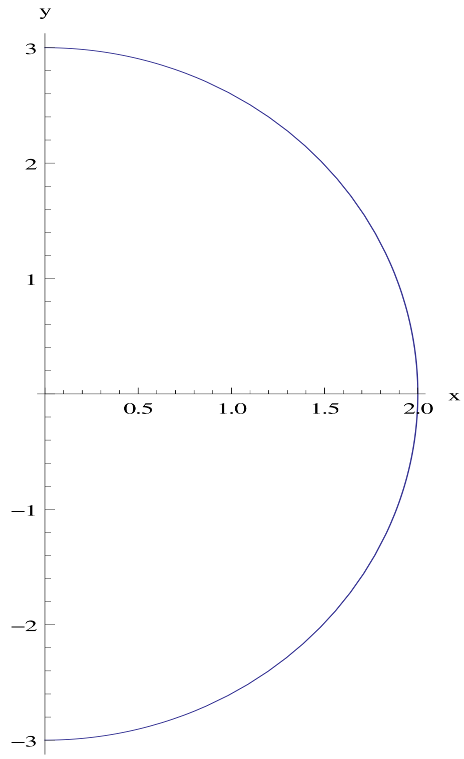

Figure 2 b: $\boldsymbol{\Phi}\,(Q)$

Submission deadline: 24.11.2023