Department of Mathematics, University of Hamburg **WiSe 2023/24**
Prof. Dr. J. Struckmeier
Dr. K. Rothe, Md. T. Hassan

Analysis III for Engineering Students

Work Sheet 2, Solutions

Exercise 1:
Given a vector field

$$\boldsymbol{g}\,(x,y) = (u(x,y), v(x,y))^T = (1, 3x^2)^T$$

a) compute $\operatorname{div}\boldsymbol{g}$ and $\operatorname{curl}\boldsymbol{g}$ and

b) make a sketch of the vector field and some streamlines in the area $[-1,1]\times[-1,1]$.

Solution:

a) $\boldsymbol{g}\,(x,y) = (u(x,y), v(x,y))^T = (1, 3x^2)^T$
$\operatorname{div}\ \boldsymbol{g} = u_x + v_y = 0\,, \quad \operatorname{rot}\ \boldsymbol{g} = v_x - u_y = 6x - 0 = 6x$

b) The MATLAB commands for plotting the vector field are

```
[X,Y] = meshgrid(-1:.2:1);
U=X.^0;
V=3*X.^2;
quiver(X,Y,U,V)
```

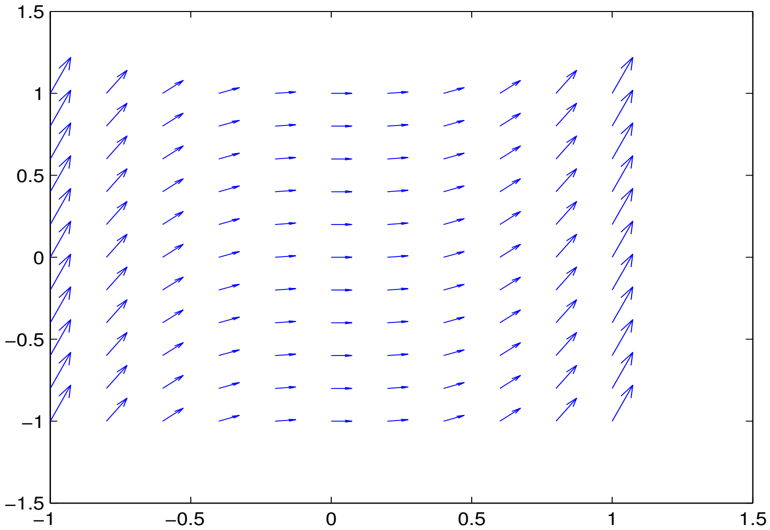

Figure 1 b.1 Vector field $\boldsymbol{g}\,(x,y) = (1, 3x^2)^T$

Streamlines are the curves $\mathbf{c}(t) = (x(t), y(t))^T$ the tangent vectors of which are given by the vector field $\boldsymbol{g}$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \boldsymbol{g}\,(x(t), y(t)) = \begin{pmatrix} u(x(t), y(t)) \\ v(x(t), y(t)) \end{pmatrix} = \begin{pmatrix} 1 \\ 3(x(t))^2 \end{pmatrix}$$

$$\Rightarrow \quad \mathbf{c}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t + a \\ (t+a)^3 + b \end{pmatrix} \overset{x(0)=0 \Rightarrow a=0}{\Rightarrow} \mathbf{c}(x) = \begin{pmatrix} x \\ x^3 + b \end{pmatrix}$$

Alternatively $y'(x) = \dfrac{v(x, y(x))}{u(x, y(x))}$: $\quad y'(x) = \dfrac{3x^2}{1} \Rightarrow y(x) = x^3 + b\,, \quad b \in \mathbb{R}$

The MATLAB commands for plotting the streamlines are:

```
[X,Y] = meshgrid(-1:.2:1);
Z = X.^3-Y;
contour(X,Y,Z)
```

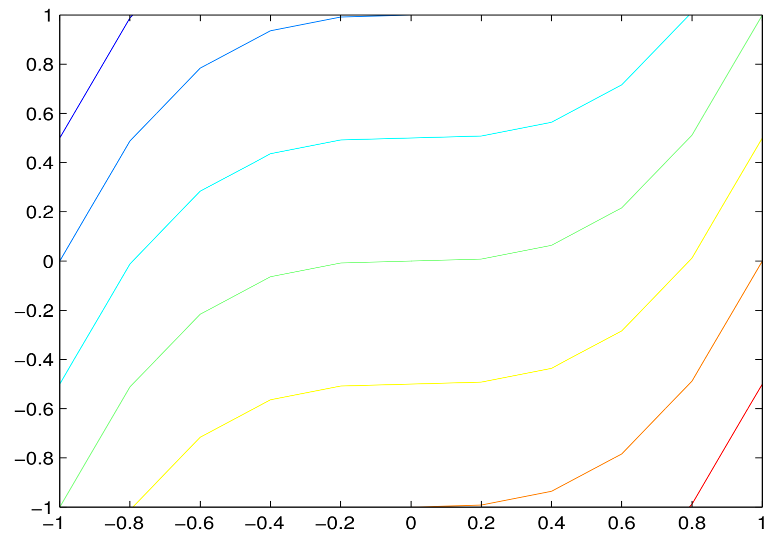

Figure 1 b.2 Streamlines $\mathbf{c}(x) = (x, x^3 + b)^T$, $b \in \mathbb{R}$

Exercise 2:

Compute the Jacobian matrices of the following functions using the definition

a) $f(x,y,z) = \sqrt{z}\sin(x+y) + e^{y+z}$ and $x, y \in \mathbb{R}$, $z \in \mathbb{R}^+$,

b) $\boldsymbol{g}(t) = (\cos t, \sin t)^T$ and $t \in \mathbb{R}$,

c) $\boldsymbol{h}(x,y) = (x+y^2, 3x^2+4y)^T$ and $x, y \in \mathbb{R}$,

d) $\boldsymbol{u}(t,x,y,z) = (x - e^{y-t}, 3z - xt^2, t + 5x + y^2 + 4z)^T$ and $t, x, y, z \in \mathbb{R}$.

Solution:

a) $f(x,y,z) = \sqrt{z}\sin(x+y) + e^{y+z}$

$$\boldsymbol{J}\, f(x,y,z) = (f_x, f_y, f_z) = \operatorname{grad} f(x,y,z)$$
$$= \left(\sqrt{z}\cos(x+y), \sqrt{z}\cos(x+y) + e^{y+z}, \frac{1}{2\sqrt{z}}\sin(x+y) + e^{y+z}\right)$$

b) $\boldsymbol{g}(t) = (\cos t, \sin t)^T$

$$\boldsymbol{J}\ \boldsymbol{g}(t) = (g_1'(t), g_2'(t))^T = \boldsymbol{g}'(t) = (-\sin t, \cos t)^T$$

c) $\boldsymbol{h}(x,y) = (x+y^2, 3x^2+4y)^T$ and $x, y \in \mathbb{R}$,

$$\boldsymbol{J}\ \boldsymbol{h}(x,y) = \begin{pmatrix} h_{1x} & h_{1y} \\ h_{2x} & h_{2y} \end{pmatrix} = \begin{pmatrix} 1 & 2y \\ 6x & 4 \end{pmatrix}$$

d) $\boldsymbol{u}(t,x,y,z) = (x - e^{y-t}, 3z - xt^2, t + 5x + y^2 + 4z)^T$ and $t, x, y, z \in \mathbb{R}$,

$$\boldsymbol{J}\ \boldsymbol{u}(t,x,y,z) = \begin{pmatrix} u_{1t} & u_{1x} & u_{1y} & u_{1z} \\ u_{2t} & u_{2x} & u_{2y} & u_{2z} \\ u_{3t} & u_{3x} & u_{3y} & u_{3z} \end{pmatrix} = \begin{pmatrix} e^{y-t} & 1 & -e^{y-t} & 0 \\ -2xt & -t^2 & 0 & 3 \\ 1 & 5 & 2y & 4 \end{pmatrix}$$

Discussion: 6.11. - 10.11.23