Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Homework sheet 2, Solutions

Exercise 1:

Compute the divergence and rate of rotation (curl) for the following vector fields with $x, y, z \in \mathbb{R}$

a) $\boldsymbol{f}\,(x,y) = (\sin x \cos y, (x+y)^2)^T$,

b) $\boldsymbol{g}\,(x,y) = (\sin y \cos x, -2xy)^T$,

c) $\boldsymbol{f}\,(x,y) + \boldsymbol{g}\,(x,y)$,

d) $\boldsymbol{h}\,(x,y,z) = (e^{x+y+z}, e^{x+y+z}, e^{x+y+z})^T$,

e) $\boldsymbol{u}\,(x,y,z) = (yz, xz, xy)^T$,

f) $2\,\boldsymbol{h}\,(x,y,z) - \boldsymbol{u}\,(x,y,z)$.

Solution:

a) $\operatorname{div} \boldsymbol{f} = f_{1x} + f_{2y} = \cos x \cos y + 2(x+y)$
$\operatorname{curl} \boldsymbol{f} = f_{2x} - f_{1y} = 2(x+y) + \sin x \sin y$

b) $\operatorname{div} \boldsymbol{g} = g_{1x} + g_{2y} = -\sin y \sin x - 2x$
$\operatorname{curl} \boldsymbol{g} = g_{2x} - g_{1y} = -2y - \cos y \cos x$

c) $\operatorname{div}(\boldsymbol{f} + \boldsymbol{g}) = \operatorname{div} \boldsymbol{f} + \operatorname{div} \boldsymbol{g}$
$= \cos x \cos y + 2(x+y) - \sin y \sin x - 2x = \cos(x+y) + 2y$
$\operatorname{curl}(\boldsymbol{f} + \boldsymbol{g}) = \operatorname{curl} \boldsymbol{f} + \operatorname{curl} \boldsymbol{g}$
$= 2(x+y) + \sin x \sin y - 2y - \cos y \cos x = 2x - \cos(x+y)$

alternatively:

$\boldsymbol{f}(x,y) + \boldsymbol{g}(x,y) = (\sin x \cos y + \sin y \cos x, (x+y)^2 - 2xy)^T$

$= (\sin(x+y), x^2+y^2)^T$

$\operatorname{div}(\boldsymbol{f} + \boldsymbol{g}) = \cos(x+y) + 2y$

$\operatorname{curl}(\boldsymbol{f} + \boldsymbol{g}) = 2x - \cos(x+y)$

d) $\boldsymbol{h}(x,y,z) = (e^{x+y+z}, e^{x+y+z}, e^{x+y+z})^T$,

$\operatorname{div}\boldsymbol{h} = h_{1x} + h_{2y} + h_{3z} = e^{x+y+z} + e^{x+y+z} + e^{x+y+z} = 3e^{x+y+z}$

$\operatorname{curl}\boldsymbol{h} = (h_{3y} - h_{2z}, h_{1z} - h_{3x}, h_{2x} - h_{1y})^T$

$= (e^{x+y+z} - e^{x+y+z}, e^{x+y+z} - e^{x+y+z}, e^{x+y+z} - e^{x+y+z})^T = \mathbf{0}$

alternatively:

$$\boldsymbol{h}(x,y,z) = \varphi(x,y,z)\,\boldsymbol{v} \quad \text{with} \quad \varphi(x,y,z) = e^{x+y+z} \quad \text{and} \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We obtain

$\nabla\varphi = \boldsymbol{h}$, $\operatorname{div}\boldsymbol{v} = 0$, $\operatorname{curl}\boldsymbol{v} = \mathbf{0}$ and

$(\nabla\varphi) \times \boldsymbol{v} = \boldsymbol{h} \times \boldsymbol{v} = (\varphi(x,y,z)\,\boldsymbol{v}) \times \boldsymbol{v} = \varphi(x,y,z)(\boldsymbol{v} \times \boldsymbol{v}) = \mathbf{0}$.

Hence

$\operatorname{div}\boldsymbol{h} = (\nabla\varphi, \boldsymbol{v}) + \varphi \operatorname{div}\boldsymbol{v} = 3e^{x+y+z}$

$\operatorname{curl}\boldsymbol{h} = (\nabla\varphi) \times \boldsymbol{v} + \varphi \operatorname{curl}\boldsymbol{v} = \mathbf{0}$.

e) $\boldsymbol{u}(x,y,z) = (yz, xz, xy)^T$,

$\operatorname{div}\boldsymbol{u} = u_{1x} + u_{2y} + u_{3z} = 0 + 0 + 0 = 0$

$\operatorname{curl}\boldsymbol{u} = (u_{3y} - u_{2z}, u_{1z} - u_{3x}, u_{2x} - u_{1y})^T = (x - x, y - y, z - z) = \mathbf{0}$

f) $\operatorname{div}(2\boldsymbol{h}(x,y,z) - \boldsymbol{u}(x,y,z)) = 2\operatorname{div}\boldsymbol{h} - \operatorname{div}\boldsymbol{u} = 6e^{x+y+z}$

$\operatorname{curl}(2\boldsymbol{h}(x,y,z) - \boldsymbol{u}(x,y,z)) = 2\operatorname{curl}\boldsymbol{h} - \operatorname{curl}\boldsymbol{u} = \mathbf{0}$

Exercise 2:

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{2x^2+3y^2} & ,\text{if} \quad (x,y) \neq (0,0) \\ 0 & ,\text{if} \quad (x,y) = (0,0)\,. \end{cases}$$

a) Draw the function in the domain $[-1,1] \times [-1,1]$.

b) Compute all directional derivatives of f at the point $(x_0, y_0) = (0,0)$.

c) Check if f is (completely) differentiable at the point $(x_0, y_0) = (0,0)$.

Solution:

a)

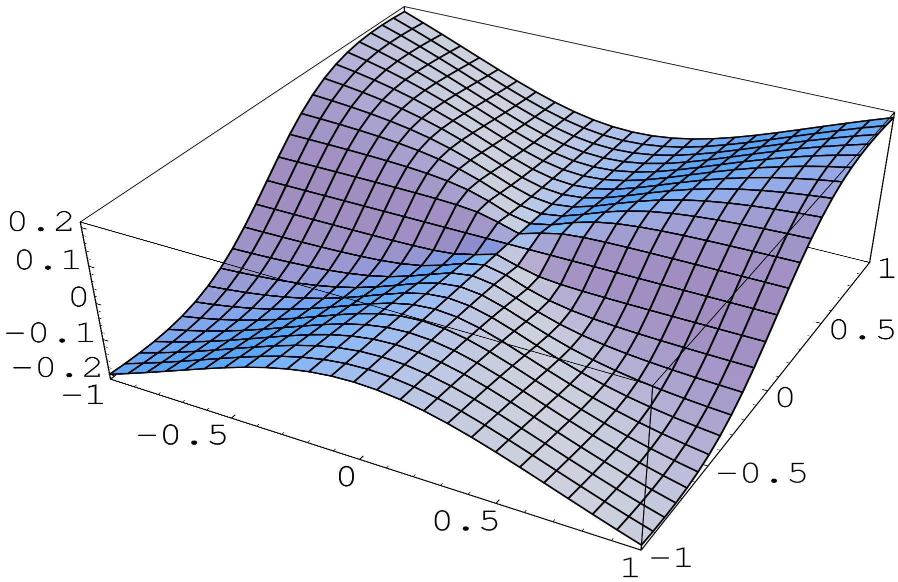

Figure 2: $f(x,y) = \dfrac{x^2 y}{2x^2+3y^2}$

b) Since we have zero in the denominator of function f at point $(x_0, y_0) = (0,0)$, we cannot assume any differentiability there, i.e. the derivatives in direction $\boldsymbol{h} = (h_1, h_2)^T$ must be computed by directly using the definition.

$$D_{\boldsymbol{h}} f(0,0) = \lim_{t\to 0} \frac{f(th_1, th_2) - f(0,0)}{t} = \lim_{t\to 0} \frac{(th_1)^2(th_2)}{t(2(th_1)^2 + 3(th_2)^2)} = f(h_1, h_2)$$

c) If f were (completely) differentiable at point $(x_0, y_0) = (0,0)$, there would have existed a linear operator $\boldsymbol{A}$ with

$$\lim_{\boldsymbol{x} \to \boldsymbol{0}} \frac{f(\boldsymbol{x}) - f(\boldsymbol{0}) - \boldsymbol{A}\,\boldsymbol{x}}{||\,\boldsymbol{x}\,||} = 0 \,.$$

Since the partial derivatives are special directional derivatives, it would hold

(ii) $\quad \boldsymbol{A} = \boldsymbol{J}\,f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = (0,0)\,.$

Now, take for example the null sequence $\boldsymbol{x}_n = (1/n, 1/n)^T$

$$\lim_{n\to\infty} \frac{f(\boldsymbol{x}_n) - f(\boldsymbol{0}) - \boldsymbol{A}\,\boldsymbol{x}_n}{||\,\boldsymbol{x}_n||} = \lim_{n\to\infty} \frac{1/n^3}{5/n^2\sqrt{2/n^2}} = \frac{1}{5\sqrt{2}}\,.$$

So f is not differentiable at the origin.

Submission deadline: 10.11.2023