Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Work sheet 1, Solutions

Exercise 1:

Compute the gradients of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$

a) $f(x, y) = x^2 + 4y^2$, b) $f(x, y) = x^2 - 4y$, c) $f(x, y) = x^2 - 4y^2$,
d) $f(x, y) = x - 4y$

and draw a graph of different contour lines of functions in the area $[-2, 2] \times [-2, 2]$. These are the lines for which $f(x, y) = c$ with $c \in \mathbb{R}$ holds.

Solution:

a) $f(x, y) = x^2 + 4y^2 \quad \Rightarrow \quad \text{grad}\, f(x, y) = (2x, 8y)$

The MATLAB command for the contour line plot is:

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ezcontour('x^2 + 4*y^2',[-2,2,-2,2])
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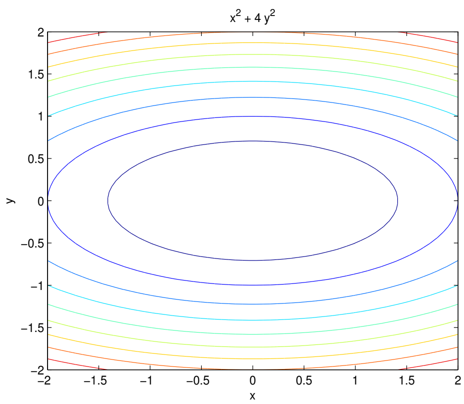

Figure 1 a) $x^2 + 4y^2 = c$ with $c \in \mathbb{R}$

b) $f(x,y) = x^2 - 4y \quad \Rightarrow \quad \operatorname{grad} f(x,y) = (2x, -4)$

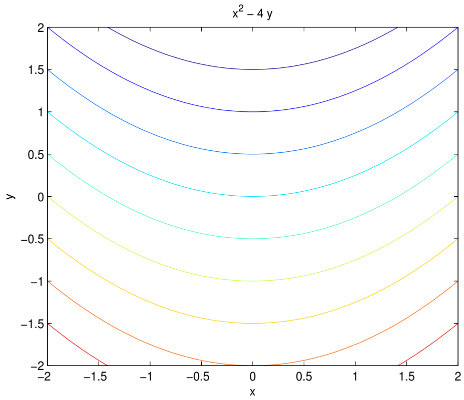

Figure 1 b) $x^2 - 4y = c$ with $c \in \mathbb{R}$

c) $f(x,y) = x^2 - 4y^2 \quad \Rightarrow \quad \operatorname{grad} f(x,y) = (2x, -8y)$

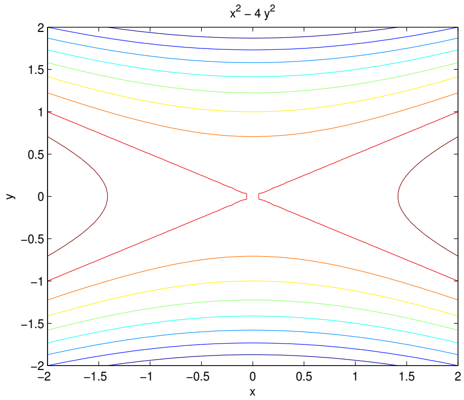

Figure 1 c) $x^2 - 4y^2 = c$ with $c \in \mathbb{R}$

d) $f(x,y) = x - 4y \quad \Rightarrow \quad \operatorname{grad} f(x,y) = (1, -4)$

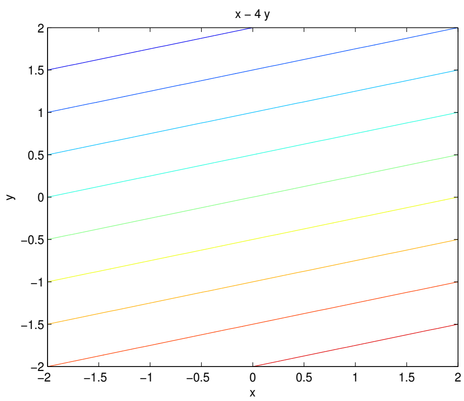

Figure 1 d) $x - 4y = c$ with $c \in \mathbb{R}$

Exercise 2:

a) Show that for a space variable x the solution to wave equation $u_{tt} = c^2 u_{xx}$ with a constant $c \in \mathbb{R}$ is given by the function

$$u(x,t) = 2\sin(x+ct) + 3e^{x-ct}.$$

b) Show that the function

$$u(x,y) = e^{-x}\sin y + (x+5)(y-6)$$

solves the Laplace equation $\Delta u = 0$.

Solution:

a) $u(x,t) = 2\sin(x+ct) + 3e^{x-ct}$

$u_t(x,t) = 2c\cos(x+ct) - 3ce^{x-ct}$,

$u_x(x,t) = 2\cos(x+ct) + 3e^{x-ct}$,

$u_{tt}(x,t) = -2c^2\sin(x+ct) + 3c^2e^{x-ct}$,

$u_{xx}(x,t) = -2\sin(x+ct) + 3e^{x-ct}$,

Hence u solves the wave equation $u_{tt} = c^2\Delta u$.

b) $u(x,y) = e^{-x}\sin y + (x+5)(y-6)$

$u_x(x,y) = -e^{-x}\sin y + y - 6$,

$u_y(x,y) = e^{-x}\cos y + x + 5$

$u_{xx}(x,y) = e^{-x}\sin y$,

$u_{yy}(x,y) = -e^{-x}\sin y$

Therefore u solves Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$.

Discussion: 23.10. - 27.10.23