Department of Mathematics, University of Hamburg **WiSe 2023/24**
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Analysis III for Engineering Students

Homework sheet 1, Solutions

Exercise 1:

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = 5x^2 - 3y^2$.

a) Compute all partial derivatives of f up to the 3rd order.

b) Visualize the graph of f in the area $[-3,3] \times [-4,4]$.

c) A tangent plane to the graph of a differentiable function f at the point $(x_0, y_0) \in D \subset \mathbb{R}^2$ is given by

$$z = z(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

Compute the tangent plane for the given function f at the point $(x_0, y_0) = (3, -4)$.

d) Give a parametric representation of the contour line of f that goes through the point $(3, -4)$.

e) Compute the angle α between grad $f(3,-4)$ and the tangential direction of the contour line of f at point $(3,-4)$.

Lösung:

a) $f(x,y) = 5x^2 - 3y^2 , \quad f_x(x,y) = 10x , \quad f_y(x,y) = -6y ,$

$f_{xx}(x,y) = 10 , \quad f_{xy}(x,y) = 0 , \quad f_{yy}(x,y) = -6 ,$

$f_{xxx}(x,y) = 0 , \quad f_{xxy}(x,y) = 0 , \quad f_{xyy}(x,y) = 0 , \quad f_{yyy}(x,y) = 0$

b) A MATLAB command for the area plot is:

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ezsurf('5*x^2-3*y^2',[-3,3,-4,4])
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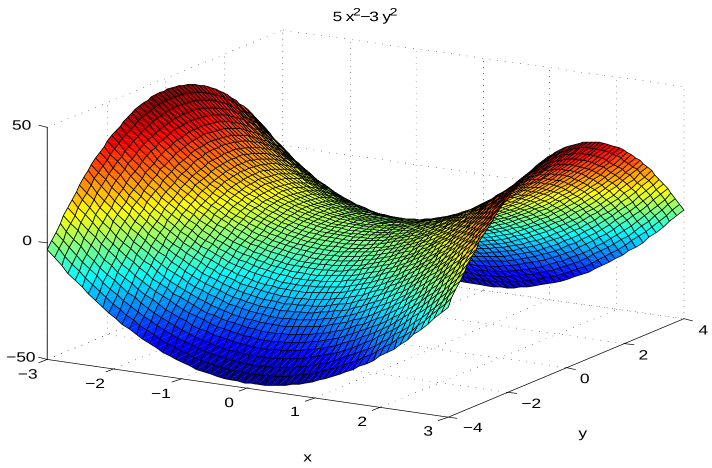

Figure 1 $f(x,y) = 5x^2 - 3y^2$

c) $f(3,-4) = 5 \cdot 3^2 - 3(-4)^2 = -3\,,\quad f_x(3,-4) = 30\,,\quad f_y(3,-4) = 24$

Tangent plane : $\quad z = -3 + 30(x-3) + 24(y+4)$

d) It is $f(3,-4) = -3$. Hence the contour line at the point $(3,-4)$ is given by the implicit equation

$$-3 = f(x, y(x)) = 5x^2 - 3y^2(x)\,.$$

Solving this gives $y(x) = \pm\sqrt{5x^2/3+1}\,.$

Since $y(3) = -4$, only $y(x) = -\sqrt{5x^2/3+1}$ is possible.

A curve parameterizing the contour line is therefore given by

$$\mathbf{c}(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix} = \begin{pmatrix} x \\ -\sqrt{5x^2/3+1} \end{pmatrix}\,.$$

e) $\operatorname{grad} f(3,-4) = (f_x(3,-4), f_y(3,-4)) = (30, 24)$

Tangential direction of the contour line

$$\mathbf{c}'(x) = \begin{pmatrix} 1 \\ -\dfrac{10x}{6\sqrt{5x^2/3+1}} \end{pmatrix} \quad\Rightarrow\quad \mathbf{c}'(3) = \begin{pmatrix} 1 \\ -\dfrac{30}{24} \end{pmatrix}$$

$$\cos\alpha = \frac{\operatorname{grad} f(3,-4)\cdot \mathbf{c}'(3)}{||\operatorname{grad} f(3,-4)||_2\, ||\mathbf{c}'(3)||_2} = 0 \quad\Rightarrow\quad \alpha = 90^\circ$$

Exercise 2:

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^4+y^4} & \text{, if } \quad (x,y) \neq (0,0) \\ 0 & \text{, if } \quad (x,y) = (0,0) \,. \end{cases}$$

a) Check if f is continuous at the origin.

b) Visualize the graph of f over the parameter domain $[-1,1] \times [-1,1]$.

c) Compute the first partial derivatives of f and

d) check whether they are continuous at the origin.

Lösung:

a) Consider the null sequence $\left(\frac{1}{k}, \frac{1}{k}\right)$ with $k \in \mathbb{N}$. It holds

$$\lim_{k\to\infty} f\left(\frac{1}{k}, \frac{1}{k}\right) = \lim_{k\to\infty} \frac{1/k^4}{2/k^4} = \frac{1}{2} \neq 0 \,.$$

The function f is therefore not continuous at zero.

b)

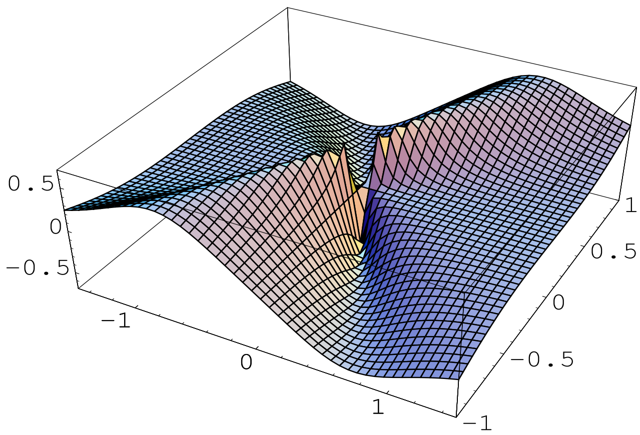

Figure 2: $\quad f(x,y) = \dfrac{xy^3}{x^4+y^4}$

c) For $(x,y) \neq (0,0)$ it holds:

$$f_x(x,y) = \frac{-3x^4y^3 + y^7}{(x^4+y^4)^2} \,, \qquad f_y(x,y) = \frac{3x^5y^2 - xy^6}{(x^4+y^4)^2}$$

for $(x,y) = (0,0)$ it holds:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0\,,$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

d) Consider the null sequence $\left(\frac{1}{k}, \frac{1}{k}\right)$ with $k \in \mathbb{N}$ to check if the partial derivatives in the zero point are continuous.

$$\lim_{k\to\infty} f_x\left(\frac{1}{k}, \frac{1}{k}\right) = \lim_{k\to\infty} \frac{-3/k^7 + 1/k^7}{(1/k^4 + 1/k^4)^2} = \lim_{k\to\infty} \frac{-k}{2} = -\infty$$

$$\lim_{k\to\infty} f_y\left(\frac{1}{k}, \frac{1}{k}\right) = \lim_{k\to\infty} \frac{3/k^7 - 1/k^7}{(1/k^4 + 1/k^4)^2} = \lim_{k\to\infty} \frac{k}{2} = \infty$$

This means that the partial derivatives are not continuous at the origin.

Submission deadline: 27.10.23