

Analysis III: Auditorium Exercise-07
For Engineering Students

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January 22, 2024

Let $\mathbf{f} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector-valued and continuous function defined as

$$\begin{aligned}\mathbf{f} : D \subset \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x := (x_1, \dots, x_n)^T &\mapsto \mathbf{f}(x_1, \dots, x_n)\end{aligned}$$

and let $c : [a, b] \rightarrow D$, $t \mapsto c(t)$ be a piecewise C^1 curve.

Definition:

$$\int_c \mathbf{f}(x) dx := \int_a^b \langle \mathbf{f}(c(t)), \dot{c}(t) \rangle dt$$

is called **Line Integral of the Second Kind**. If the curve is **closed**, i.e., $c(a) = c(b)$, one can also write $\oint_c \mathbf{f}(x) dx$.



Potential Calculation:

A vector field $\mathbf{f} : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ possesses a **potential** or an **antiderivative** if there exists a C^1 function $\Phi : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that \mathbf{f} coincides with the **gradient field** of Φ :

$$\text{grad } \Phi(\mathbf{x}) = \mathbf{f}(\mathbf{x}) .$$

A C^1 vector field $\mathbf{f} : D \rightarrow \mathbb{R}^n$ possesses a potential in a simply connected region $D \subset \mathbb{R}^n$ if and only if the following integrability condition is satisfied for all $\mathbf{x} \in D$:

$$\mathbf{Jf}(\mathbf{x}) = (\mathbf{Jf}(\mathbf{x}))^T .$$

For $n = 2, 3$, this condition coincides with $\text{rot } \mathbf{f}(\mathbf{x}) = \mathbf{0}$.

If there exists a potential for the vector field \mathbf{f} , then its called a **Conservative** field.



For a continuous vector field $\mathbf{f} : D \rightarrow \mathbb{R}^n$ with potential Φ , the following holds:

$$(a) \quad \int_C \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = \Phi(\mathcal{C}(b)) - \Phi(\mathcal{C}(a))$$

for any piecewise C^1 curve $\mathcal{C} : [a, b] \rightarrow D$.

(b) A potential Φ associated with \mathbf{f} can be calculated by

$$\Phi(\mathbf{x}) = \int_{\mathcal{C}_x} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} + \text{Constant} .$$

Here, \mathcal{C}_x is any piecewise C^1 curve in D connecting a fixed point $\mathbf{x}_0 \in D$ to $\mathbf{x} \in D$.



Another way to calculate a potential (in addition to b) is by successively **'integrating'** the components of the vector field

$$\mathbf{f} = (f_1, f_2, f_3)^T,$$

using the condition $\mathbf{grad} \Phi(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, so in \mathbb{R}^3 :

$$\begin{pmatrix} \Phi_x(x, y, z) \\ \Phi_y(x, y, z) \\ \Phi_z(x, y, z) \end{pmatrix} = \begin{pmatrix} f_1(x, y, z) \\ f_2(x, y, z) \\ f_3(x, y, z) \end{pmatrix}.$$



Consider the vector field $\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$\mathbf{f}(x, y, z) = \begin{pmatrix} 3x^2y^4z^5 + 1 \\ 4x^3y^3z^5 + 2y \\ 5x^3y^4z^4 + 3z^2 \end{pmatrix}.$$

- Show that \mathbf{f} has a potential without explicitly calculating it.
- Calculate a potential by successively integrating \mathbf{f} .
- Calculate a potential using the Fundamental Theorem for Line Integrals.
- Along the curve $\mathcal{C} : [0, T] \rightarrow \mathbb{R}^3$ given by $\mathcal{C}(t) = (\cos t, \sin t, \sin t + \cos t)^T$ compute the curve integral $\int_{\mathcal{C}} \mathbf{f}(\mathbf{x})d\mathbf{x}$ for the cases $T = \pi$ and $T = 2\pi$.



(a) The space \mathbb{R}^3 is simply connected, and the integrability condition

$$\operatorname{curl} \mathbf{f}(x, y, z) = \begin{pmatrix} f_{3y} - f_{2z} \\ f_{1z} - f_{3x} \\ f_{2x} - f_{1y} \end{pmatrix} = \begin{pmatrix} 20x^3y^3z^4 - 20x^3y^3z^4 \\ 15x^2y^4z^4 - 15x^2y^4z^4 \\ 12x^2y^3z^5 - 12x^2y^3z^5 \end{pmatrix} = \mathbf{0}$$

is satisfied. Therefore, $\mathbf{f}(x, y, z)$ has a potential $v(x, y, z)$, i.e., $\mathbf{f} = \operatorname{grad} v = (v_x, v_y, v_z)$.

$$(b) \quad v_x(x, y, z) = 3x^2y^4z^5 + 1 \quad \Rightarrow \quad v(x, y, z) = x^3y^4z^5 + x + c(y, z)$$

$$\Rightarrow \quad v_y(x, y, z) = 4x^3y^3z^5 + c_y(y, z) \stackrel{!}{=} 4x^3y^3z^5 + 2y$$

$$\Rightarrow \quad c_y(y, z) = 2y \quad \Rightarrow \quad c(y, z) = y^2 + k(z)$$

$$\Rightarrow \quad v(x, y, z) = x^3y^4z^5 + x + y^2 + k(z)$$

$$\Rightarrow \quad v_z(x, y, z) = 5x^3y^4z^4 + k'(z) \stackrel{!}{=} 5x^3y^4z^4 + 3z^2$$

$$\Rightarrow \quad k'(z) = 3z^2 \quad \Rightarrow \quad k(z) = z^3 + K \quad \text{with } K \in \mathbb{R}$$

$$\Rightarrow \quad v(x, y, z) = x^3y^4z^5 + x + y^2 + z^3 + K$$



(c) Choosing the curve \mathcal{K} as the direct connection line from the point $(0, 0, 0)$ to the point (x, y, z) , i.e., $\mathcal{K}(t) = t(x, y, z)^T$, a potential v for \mathbf{f} can be calculated using the Fundamental Theorem for Line Integrals as follows:

$$\begin{aligned}v(x, y, z) &= \int_{\mathcal{K}} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} + K = \int_0^1 \mathbf{f}(\mathcal{K}(t)) \dot{\mathcal{K}}(t) \, dt + K \\&= \int_0^1 \left\langle \begin{pmatrix} 3(tx)^2(ty)^4(tz)^5 + 1 \\ 4(tx)^3(ty)^3(tz)^5 + 2ty \\ 5(tx)^3(ty)^4(tz)^4 + 3(tz)^2 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle dt + K \\&= \int_0^1 12t^{11}x^3y^4z^5 + xt + 2ty^2 + 3t^2z^3 \, dt + K \\&= t^{12}x^3y^4z^5 + xt + t^2y^2 + t^3z^3 \Big|_0^1 + K \\&= x^3y^4z^5 + x + y^2 + z^3 + K\end{aligned}$$



(d) With $\mathcal{C}(t) = (\cos t, \sin t, \sin t + \cos t)^T$, the Fundamental Theorem for Line Integrals yields

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{f}(\mathbf{x}) d\mathbf{x} &= \int_0^{\pi} \mathbf{f}(\mathcal{C}(t)) \dot{\mathcal{C}}(t) dt = v(\mathcal{C}(\pi)) - v(\mathcal{C}(0)) \\ &= v(-1, 0, -1) - v(1, 0, 1) = -1 - 1 - (1 + 1) = -4\end{aligned}$$

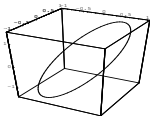


Figure: Curve \mathcal{C} for $T = 2\pi$, (closed curve)

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{f}(\mathbf{x}) d\mathbf{x} &= \int_0^{2\pi} \mathbf{f}(\mathcal{C}(t)) \dot{\mathcal{C}}(t) dt = v(\mathcal{C}(2\pi)) - v(\mathcal{C}(0)) \\ &= v(1, 0, 1) - v(1, 0, 1) = 0\end{aligned}$$

Consider a C^1 vector field $\mathbf{f} : G \rightarrow \mathbb{R}^2$ on the domain $G \subset \mathbb{R}^2$, and a compact set $D \subset G$, which is representable as a normal region with respect to both coordinate axes. Then, the following holds:

$$\int_D \operatorname{rot} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = \oint_{\partial D} \mathbf{f}(\mathbf{x}) \, d\mathbf{x}.$$

Here, the boundary ∂D , in the chosen parameterization for computation, must be traversed in a mathematically positive direction, i.e., counterclockwise.



Verify Green's Theorem for the vector field

$$\mathbf{f}(x, y) = (-xy - 2y, 2x + 4y^2)^T$$

and the region E enclosed by the curve $x^2 + 4y^2 = 4$.



The ellipse E can be described in Cartesian or polar coordinates as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \Phi(r, \varphi) = \begin{pmatrix} 2r \cos \varphi \\ r \sin \varphi \end{pmatrix}, \quad 0 \leq r \leq 1, \quad 0 \leq \varphi \leq 2\pi, \quad \Rightarrow \det \Phi(r, \varphi) = 2r$$

$$E = \left\{ (x, y)^T \in \mathbb{R}^2 \mid -2 \leq x \leq 2, -\sqrt{1 - (x/2)^2} \leq y \leq \sqrt{1 - (x/2)^2} \right\},$$

$$Q = \left\{ (r, \varphi)^T \in \mathbb{R}^2 \mid 0 \leq r \leq 1, 0 \leq \varphi \leq 2\pi \right\} \quad \text{with} \quad \Phi(Q) = E$$

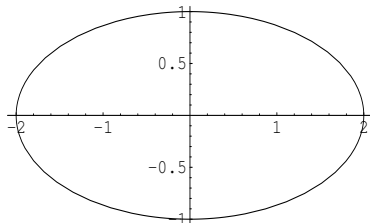


Figure: Ellipse E



Parametrization of the ellipse boundary ∂E by:

$$\mathbf{c}(\varphi) = \begin{pmatrix} 2 \cos \varphi \\ \sin \varphi \end{pmatrix}, \quad 0 \leq \varphi \leq 2\pi$$

$$\begin{aligned} \oint_{\partial E} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} &= \oint_{\mathbf{c}} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = \int_0^{2\pi} \langle \mathbf{f}(\mathbf{c}(\varphi)), \dot{\mathbf{c}}(\varphi) \rangle \, d\varphi \\ &= \int_0^{2\pi} \left\langle \begin{pmatrix} -2 \cos \varphi \sin \varphi - 2 \sin \varphi \\ 4 \cos \varphi + 4 \sin^2 \varphi \end{pmatrix}, \begin{pmatrix} -2 \sin \varphi \\ \cos \varphi \end{pmatrix} \right\rangle \, d\varphi \\ &= \int_0^{2\pi} 4 \cos \varphi \sin^2 \varphi + 4 \sin^2 \varphi + 4 \cos^2 \varphi + 4 \cos \varphi \sin^2 \varphi \, d\varphi \\ &= \int_0^{2\pi} 4 + 8 \cos \varphi \sin^2 \varphi \, d\varphi = 4\varphi + \frac{8}{3} \sin^3 \varphi \Big|_0^{2\pi} = 8\pi \end{aligned}$$



$$\begin{aligned}\int_E \operatorname{rot} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} &= \int_E (2x + 4y^2)_x - (-xy - 2y)_y \, d(x, y) \\ &= \int_E 4 + x \, d(x, y) = \int_0^1 \int_0^{2\pi} (4 + 2r \cos \varphi) 2r \, d\varphi dr \\ &= 8 \int_0^1 r \, dr \int_0^{2\pi} d\varphi + 4 \int_0^1 r^2 \, dr \int_0^{2\pi} \cos \varphi \, d\varphi \\ &= 8\pi r^2 \Big|_0^1 + \frac{4r^3}{3} \Big|_0^1 \cdot \sin \varphi \Big|_0^{2\pi} = 8\pi\end{aligned}$$

Green's Theorem: $\oint_{\partial E} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = 8\pi = \int_E \operatorname{rot} \mathbf{f}(\mathbf{x}) \, d\mathbf{x}$



Let $G \subset \mathbb{R}^2$ be a region and consider a C^1 mapping

$$p : G \rightarrow \mathbb{R}^3$$
$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto p(u) = \begin{pmatrix} x(u_1, u_2) \\ y(u_1, u_2) \\ z(u_1, u_2) \end{pmatrix}.$$

If the vectors $\frac{\partial p(u)}{\partial u_1}$ and $\frac{\partial p(u)}{\partial u_2}$ are linearly independent for all $u \in G$, then,

(a) $F := \{p(u) \in \mathbb{R}^3 \mid u \in G\} = p(G)$

Surface or **Surface Patch** in \mathbb{R}^3 .

(b) p **Parametrization** of F ,



(c) G **Parameter Range** of F with respect to p ,

(d) $T_F(\lambda, \mu) = p(u^0) + \lambda \frac{\partial p(u^0)}{\partial u_1} + \mu \frac{\partial p(u^0)}{\partial u_2}$ with $\lambda, \mu \in \mathbb{R}$ **Tangent Plane** at the point $p(u^0)$ to F ,

(e) $\frac{\partial p(u^0)}{\partial u_1} \times \frac{\partial p(u^0)}{\partial u_2}$ **Normal Vector** to F at the point $p(u^0)$,

(f) $n(p(u)) := \frac{\frac{\partial p(u)}{\partial u_1} \times \frac{\partial p(u)}{\partial u_2}}{\left\| \frac{\partial p(u)}{\partial u_1} \times \frac{\partial p(u)}{\partial u_2} \right\|}$ **Unit Normal Vector** to F ,

(g) $do := \left\| \frac{\partial p}{\partial u_1} \times \frac{\partial p}{\partial u_2} \right\|$ **Surface Element** and

(h) $\int_{p(G)} do := \int_G \left\| \frac{\partial p(u)}{\partial u_1} \times \frac{\partial p(u)}{\partial u_2} \right\| du$ **Surface Area** of $p(G)$.



For the surface F parametrized by the compact, measurable, and connected set D using the C^1 mapping p , i.e., $F = p(D)$, the following surface integrals are defined:

(a) **Surface Integral of the First Kind** for the continuous function $f : F \rightarrow \mathbb{R}$

$$\int_F f(\mathbf{x}) \, do := \int_D f(p(\mathbf{u})) \left\| \frac{\partial p(\mathbf{u})}{\partial u_1} \times \frac{\partial p(\mathbf{u})}{\partial u_2} \right\| \, d\mathbf{u} .$$



(b) **Surface Integral of the Second Kind** for the continuous vector field $\mathbf{f} : F \rightarrow \mathbb{R}^3$

$$\begin{aligned}\int_F \mathbf{f}(\mathbf{x}) \, do &:= \int_F \langle \mathbf{f}(\mathbf{x}), \mathbf{n}(\mathbf{x}) \rangle \, do \\ &= \int_D \langle \mathbf{f}(\mathbf{x}), \mathbf{n}(\mathbf{x}) \rangle \left\| \frac{\partial p}{\partial u_1} \times \frac{\partial p}{\partial u_2} \right\| \, d\mathbf{u} \\ &= \int_D \left\langle \mathbf{f}(p(\mathbf{u})), \frac{\partial p(\mathbf{u})}{\partial u_1} \times \frac{\partial p(\mathbf{u})}{\partial u_2} \right\rangle \, d\mathbf{u}.\end{aligned}$$

Note:

If the vector field \mathbf{f} represents the velocity field of a stationary flow, then the surface integral $\int_F \mathbf{f}(\mathbf{x}) \, do$ can be interpreted as the **flux** of \mathbf{f} through the surface F , measured in the amount of fluid per unit time in the direction of the chosen normal.



Evaluate

$$\iint_S y \, dS$$

where S is the portion of the cylinder $x^2 + y^2 = 3$ that lies between $z = 0$ and $z = 6$.



Parameterization:

$$\vec{p}(z, \varphi) = \sqrt{3} \cos \varphi \vec{i} + \sqrt{3} \sin \varphi \vec{j} + z \vec{k}$$

The ranges of parameters:

$$0 \leq z \leq 6, \quad 0 \leq \varphi \leq 2\pi$$

The cross product:

$$\vec{p}_z \times \vec{p}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -\sqrt{3} \sin \varphi & \sqrt{3} \cos \varphi & 0 \end{vmatrix} = -\sqrt{3} \cos \varphi \vec{i} - \sqrt{3} \sin \varphi \vec{j}$$

The magnitude of this vector is,

$$\|\vec{p}_z \times \vec{p}_\varphi\| = \sqrt{3}$$



The surface integral,

$$\begin{aligned} & \iint_S y \, dS \\ &= \iint_D \sqrt{3} \sin \varphi (\sqrt{3}) \, d(z, \varphi) \\ &= 3 \int_0^{2\pi} \int_0^6 \sin \varphi \, dz \, d\varphi \\ &= 3 \int_0^{2\pi} 6 \sin \varphi \, d\varphi = (-18 \cos \varphi) \Big|_0^{2\pi} = 0 \end{aligned}$$



For the C^1 vector field $f : G \rightarrow \mathbb{R}^3$ on the domain $G \subset \mathbb{R}^3$ and the compact measurable standard region $S \subset G$, whose boundary ∂S consists of finitely many smooth surface pieces, the following holds:

$$\int_S \operatorname{div} f(x) \, dx = \oint_{\partial S} f(x) \, do.$$

In the calculation of the surface integral over the closed surface ∂S , hence the notation $\oint_{\partial S}$ is used, the normal vector $\frac{\partial p(u)}{\partial u_1} \times \frac{\partial p(u)}{\partial u_2}$ with respect to S points outward.

Remark:

If the vector field represents the velocity field of a stationary flow, then the surface integral $\oint_{\partial S} f(x) \, do$ can be interpreted as the flow balance through the volume S . For $\operatorname{div} f(x) = 0$ in S , according to Gauss's Divergence Theorem, $\oint_{\partial S} f(x) \, do = 0$, meaning that as much flows out of S as flows into it.



Given the solid region

$$K = \{ (x, y, z)^T \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 9, x \leq 0 \}$$

and the vector field

$$f(x, y, z) = (y, -x, z^3)^T.$$

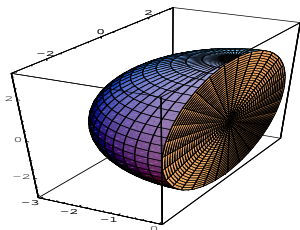
- (a) Sketch K .
- (b) The boundary of K can be described by a planar surface piece S and a non-planar surface piece H .

Provide parametrizations for both boundary surface pieces S and H .

- (c) Calculate the flux of f through both boundary surface pieces S and H .
- (d) Calculate the volume integral $\int_E \operatorname{div} (x, y, z) d(x, y, z)$.



(a)

**Figure:** Hemisphere K

(b) Parametrization of the circular side S : $p : [0, 3] \times [0, 2\pi] \rightarrow \mathbb{R}^3$ with

$$p(r, \varphi) = \begin{pmatrix} 0 \\ r \cos \varphi \\ r \sin \varphi \end{pmatrix}$$

Parametrization of the hemisphere surface H :

$$q : \left[\frac{\pi}{2}, \frac{3\pi}{2} \right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \rightarrow \mathbb{R}^3 \text{ with}$$

$$q(\varphi, \psi) = \begin{pmatrix} 3 \cos \varphi \cos \psi \\ 3 \sin \varphi \cos \psi \\ 3 \sin \psi \end{pmatrix}$$

(c) Flux through S , with the outward normal

$$\frac{\partial p}{\partial r} \times \frac{\partial p}{\partial \varphi} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -r \sin \varphi & r \cos \varphi \end{vmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}$$

$$\int_S do = \int_0^3 \int_0^{2\pi} \left\langle \begin{pmatrix} r \cos \varphi \\ 0 \\ r^3 \sin^3 \varphi \end{pmatrix}, \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \right\rangle d\varphi dr = \int_0^3 \int_0^{2\pi} r^2 \cos \varphi d\varphi dr = 0$$



Flux through H , with the outward normal

$$\frac{\partial q}{\partial \varphi} \times \frac{\partial q}{\partial \psi} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -3 \sin \varphi \cos \psi & 3 \cos \varphi \cos \psi & 0 \\ -3 \cos \varphi \sin \psi & -3 \sin \varphi \sin \psi & 3 \cos \psi \end{vmatrix} = 9 \cos \psi \begin{pmatrix} \cos \varphi \cos \psi \\ \sin \varphi \cos \psi \\ \sin \psi \end{pmatrix}$$

$$\int_H d\mathbf{o} = \int_{\pi/2}^{3\pi/2} \int_{-\pi/2}^{\pi/2} 9 \cos \psi \left\langle \begin{pmatrix} 3 \sin \varphi \cos \psi \\ -3 \cos \varphi \cos \psi \\ 27 \sin^3 \psi \end{pmatrix}, \begin{pmatrix} \cos \varphi \cos \psi \\ \sin \varphi \cos \psi \\ \sin \psi \end{pmatrix} \right\rangle d\psi d\varphi$$

$$= \int_{\pi/2}^{3\pi/2} \int_{-\pi/2}^{\pi/2} 243 \cos \psi \sin^4 \psi d\psi d\varphi = 243\pi \frac{\sin^5 \psi}{5} \Big|_{-\pi/2}^{\pi/2} = \frac{486\pi}{5}$$



(d) Using the Gaussian Integral Theorem:

$$\int_E \operatorname{div} f \, d(x, y, z) = \int_S f \, do + \int_H f \, do = \frac{486\pi}{5}$$

Alternatively: direct calculation using spherical coordinates:

$$\begin{aligned} & \int_K \operatorname{div} f(x, y, z) \, d(x, y, z) \\ = & \int_K 3z^2 \, d(x, y, z) = \int_0^3 \int_{\pi/2}^{3\pi/2} \int_{-\pi/2}^{\pi/2} 3r^2 \sin^2 \psi \cdot r^2 \cos \psi \, d\psi d\varphi dr \\ = & \int_0^3 r^4 dr \int_{\pi/2}^{3\pi/2} d\varphi \int_{-\pi/2}^{\pi/2} 3 \cos \psi \sin^2 \psi \, d\psi = \frac{r^5}{5} \Big|_0^3 \varphi \Big|_{\pi/2}^{3\pi/2} \sin^3 \psi \Big|_{-\pi/2}^{\pi/2} = \frac{486\pi}{5} \end{aligned}$$



THANK YOU

