Analysis III: Auditorium Exercise-06

For Engineering Students

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Exemplary representation for a **bounded function** on a **rectangle**

$$\begin{aligned} f : \underbrace{[a,b] \times [c,d]}_{:=Q} &\to \mathbb{R} \\ (x,y) &\mapsto f(x,y)\,. \end{aligned}$$

Partition Z of the rectangle Q by

$$a = x_0 < x_1 < \cdots < x_n = b\,, \quad c = y_0 < y_1 < \cdots < y_m = d$$

into **subrectangles**

$$Q_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$

with **area** $\mathrm{Area}(Q_{ij}) = (x_{i+1} - x_i) \cdot (y_{j+1} - y_j)$.

Riemann Lower Sum: (Untersumme)

$$U_f(Z) := \sum_{i=0}^{n-1} \left(\sum_{j=0}^{m-1} \inf_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}(Q_{i,j}) \right)$$

Riemann Upper Sum: (Obersumme)

$$O_f(Z) := \sum_{i=0}^{n-1} \left(\sum_{j=0}^{m-1} \sup_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}(Q_{i,j}) \right)$$

Riemann Integral: (defined only if $\sup_Z U_f(Z) = \inf_Z O_f(Z)$)

$$\int_Q f(x,y)\, d(x,y) \;:=\; \sup_Z U_f(Z) \qquad \left(= \inf_Z O_f(Z) \right) .$$

Theorem: (Fubini's Theorem)

If

$$F(x) := \int_c^d f(x,y)\,dy$$

exists for all $x \in [a,b]$ and

$$G(y) := \int_a^b f(x,y)\,dx$$

exists for all $y \in [c,d]$, then

$$\int\limits_Q f(x,y)\,d(x,y) = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy\,.$$

For $Q := [0,2] \times [0,1]$, compute for the function

$$f : Q \to \mathbb{R}\,, \quad f(x,y) = 2 - x$$

(a) Compute the Riemann lower and upper sums for the following partition Z of Q

$$Q_{i,j} = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]\,, \; i,j = 1,\ldots,n$$

(b) Compute the integral of f over Q according to Fubini's theorem.

Example: 01 (a)

Riemann lower and upper sums:

$$\begin{aligned}
U_f(Z) &= \textstyle\sum_{i,j=1}^{n} \inf_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}\,(Q_{i,j}) \\
&= \textstyle\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(2 - \frac{2i}{n}\right) \cdot \frac{2}{n^2} \right) \\
&= \textstyle\frac{4}{n^2} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(1 - \frac{i}{n}\right) \right) \\
&= \textstyle\frac{4}{n^2} \sum_{i=1}^{n} (n - i) \\
&= \textstyle\frac{2(n^2-n)}{n^2} \;=\; 2\left(1 - \frac{1}{n}\right)
\end{aligned}$$

$$\begin{aligned}
O_f(Z) &= \textstyle\sum_{i,j=1}^{n} \sup_{(x,y)\in Q_{i,j}} (f(x,y)) \cdot \text{Area}\,(Q_{i,j}) \\
&= \textstyle\sum_{i=1}^{n} \left(\sum_{j=1}^{n} \left(2 - \frac{2(i-1)}{n}\right) \cdot \frac{2}{n^2} \right) \;=\; 2\left(1 + \frac{1}{n}\right)
\end{aligned}$$

The integral of f over Q according to Fubini's theorem.

$$\begin{aligned}
\int_Q f(x,y)\, d(x,y) &= \int_0^1 \left(\int_0^2 2 - x \, dx \right) dy \\
&= \int_0^1 2x - \frac{x^2}{2} \Big|_0^2 \, dy \\
&= \int_0^1 2 \, dy \\
&= 2y\big|_0^1 \; = \; 2
\end{aligned}$$

Of course, one obtains:

$$2\left(1 - \frac{1}{n}\right) = U_f(Z) \;\leq\; \int_Q f(x,y)\, d(x,y) = 2$$

$$\int_Q f(x,y)\, d(x,y) = 2 \;\leq\; O_f(Z) = 2\left(1 + \frac{1}{n}\right) \; .$$

Compute the following integrals:

(a) $\int_{\pi}^{2\pi} \int_{0}^{\pi} \cos(x + y)\, dx\, dy$,

(b) $\int_R 9x^2 \sqrt{y}\, d(x, y)$ with $R = [1, 2] \times [1, 4]$,

(c) $\int_Q \sinh z + \frac{6z^2}{(2x+y)^2}\, d(x, y, z)$ with $Q = [1, 2] \times [0, 1] \times [-1, 1]$.

(a)

$$\int_{\pi}^{2\pi}\int_0^{\pi} \cos(x+y)\,dx\,dy = \int_{\pi}^{2\pi} \sin(x+y)|_0^{\pi}\,dy$$
$$= \int_{\pi}^{2\pi} \sin(\pi+y) - \sin y\,dy$$
$$= (\cos y - \cos(\pi+y)|_{\pi}^{2\pi} = 4$$

(b)

$R = [1,2]\times[1,4]$,

$$\int_R 9x^2\sqrt{y}\,d(x,y) = \int_1^2\int_1^4 9x^2\sqrt{y}\,dy\,dx$$
$$= \int_1^2 3x^2\left(\int_1^4 3\sqrt{y}\,dy\right)\,dx$$
$$= \left(\int_1^2 3x^2\,dx\right)\cdot\left(\int_1^4 3\sqrt{y}\,dy\right)$$
$$= \left(x^3\Big|_1^2\right)\cdot\left(2y^{3/2}\Big|_1^4\right) = 98$$

Example: 02 (c)

$Q = [1,2] \times [0,1] \times [-1,1]$.

$$
\begin{aligned}
& \int_Q \sinh z + \tfrac{6z^2}{(2x+y)^2}\, d(x,y,z) \\
= \quad & \int_1^2 \int_0^1 \int_{-1}^1 \sinh z + \tfrac{6z^2}{(2x+y)^2}\, dz\, dy\, dx \\
= \quad & \int_1^2 \int_0^1 \left(\cosh z + \tfrac{2z^3}{(2x+y)^2}\right)\Big|_{-1}^{1}\, dy\, dx \\
= \quad & \int_1^2 \int_0^1 \tfrac{4}{(2x+y)^2}\, dy\, dx = \int_1^2 -\tfrac{4}{2x+y}\Big|_0^1\, dx \\
= \quad & \int_1^2 -\tfrac{4}{2x+1} + \tfrac{2}{x}\, dx = (-2\ln|2x+1| + 2\ln|x|)\big|_1^2 \\
= \quad & -2\ln 5 + 2\ln 2 + 2\ln 3 = \ln \tfrac{36}{25}
\end{aligned}
$$

Normal regions in $\mathbb{R}^2$

A set $D \subset \mathbb{R}^2$ is called a **normal region** if

1. continuous functions $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ exist such that D has the following representation:

$$D = \{\, (x, y) \mid a \leq x \leq b\,,\ \varphi_1(x) \leq y \leq \varphi_2(x) \,\}$$

2. continuous functions $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ exist such that D has the following representation:

$$D = \{\, (x, y) \mid \psi_1(y) \leq x \leq \psi_2(y)\,,\ c \leq y \leq d \,\}\ .$$

Normal regions in $\mathbb{R}^3$

A set $D \subset \mathbb{R}^3$ is called a **normal region** if continuous functions φ_1, φ_2 and ξ_1, ξ_2 exist, such that D has the following representation:

$$D = \{ (x, y, z) \mid a \leq x \leq b \, , \, \varphi_1(x) \leq y \leq \varphi_2(x) \, , \, \xi_1(x, y) \leq z \leq \xi_2(x, y) \}$$

As in the representation in $\mathbb{R}^2$, x, y, and z can be arbitrarily interchanged.

Remark:
Often, sets D over which integration is to be performed cannot be represented by a single normal region, but only by the union of several normal regions.

Example: 03

1. 1.1 Draw the triangle D with vertices $P_1 = (-1, 1)$, $P_2 = (0, 0)$ and $P_3 = (2, 2)$ and represent it as a normal region.

   1.2 Calculate $\int_D 18y \, d(x, y)$

2. 2.1 Draw the region Z described by $x \leq 0$, $z \geq 1$, $z \leq 3$, and $x^2 + y^2 \leq 4$, and represent it as a normal region.

   2.2 Calculate $\int_Z 3x \, d(x, y, z)$

The lines through the given points are:
P_1, P_3: $g(x) = (x+4)/3$, P_1, P_2: $f_1(x) = -x$, P_2, P_3: $f_2(x) = x$.

$$D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; -1 \le x \le 2 \,,\; |x| \le y \le (x+4)/3 \right\}$$

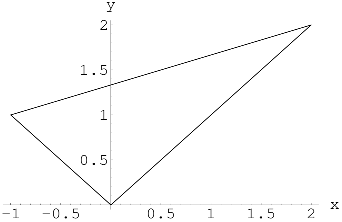

Image : Triangle D

Example: 03 (1.2)

$$\int_D 18y\, d(x,y) = \int_{-1}^{2} \int_{|x|}^{(x+4)/3} 18y\, dy\, dx = \int_{-1}^{2} 9y^2 \Big|_{|x|}^{(x+4)/3} dx$$

$$= \int_{-1}^{2} (x+4)^2 - 9x^2\, dx = \frac{(x+4)^3}{3} - 3x^3 \Big|_{-1}^{2} = 36$$

$x \le 0,\ z \ge 1,\ z \le 3$ and $x^2 + y^2 \le 4$ describes a half cylinder

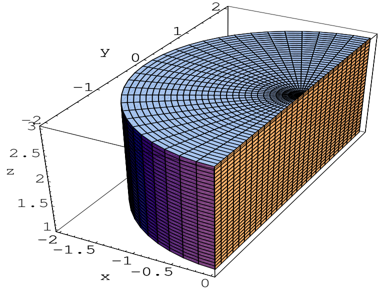

Image : Half cylinder Z

$$Z = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, -2 \le x \le 0\,,\ -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}\,,\ 1 \le z \le 3 \right\}$$

Example: 03 (2.2)

$$\int_Z 3x\, d(x,y,z) = \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{1}^{3} 3x\, dz\, dy\, dx = \int_{1}^{3} dz \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3x\, dy\, dx$$

$$= 2 \int_{-2}^{0} 3xy\Big|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = 2 \int_{-2}^{0} 6x\sqrt{4-x^2}\, dx$$

$$= -4\,(4-x^2)^{3/2}\Big|_{-2}^{0} = -32$$

Example: 03 (2.2)

or alternatively with transformation to cylindrical coordinates:

$$\begin{aligned}
\int_Z 3x \, d(x,y,z) &= \int_1^3 \int_{\pi/2}^{3\pi/2} \int_0^2 3r\cos(\varphi) r \, dr \, d\varphi \, dz \\
&= \int_0^2 3r^2 \, dr \int_{\pi/2}^{3\pi/2} \cos(\varphi) \, d\varphi \int_1^3 \, dz \\
&= \left(r^3 \Big|_0^2 \right) \left(\sin(\varphi) \Big|_{\pi/2}^{3\pi/2} \right) \left(z \Big|_1^3 \right) \\
&= 8 \cdot (-2) \cdot 2 = -32
\end{aligned}$$

Consider a body $K \subset \mathbb{R}^3$ with the nonnegative continuous mass density function $\rho : K \to \mathbb{R}$.

The **mass** M of the body K is calculated by

$$M = \int_K \rho(x, y, z)\, d(x, y, z)\,.$$

The **center of mass** $\mathbf{x}_s$ of the body K is given by

$$\mathbf{x}_s = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \int_K \rho(x, y, z) \cdot x\, d(x, y, z) \\ \int_K \rho(x, y, z) \cdot y\, d(x, y, z) \\ \int_K \rho(x, y, z) \cdot z\, d(x, y, z) \end{pmatrix}.$$

The **moment of inertia** Θ_A of a body K with respect to an axis A is calculated by

$$\Theta_A = \int_K \rho(x,y,z) r^2(x,y,z)\, d(x,y,z)\,.$$

Here, $r(x,y,z)$ represents the distance of the point $(x,y,z)^T \in K$ to A.

Steiner's Theorem:
If S is an axis parallel to A and passing through the center of mass $\mathbf{x}_s$ of the body K, d is the distance of the axis A from $\mathbf{x}_s$, and M is the mass of K, then, for constant density ρ, the following holds

$$\Theta_A = Md^2 + \Theta_S.$$

1. **Polar Coordinates:** $0 \le r \le R\,, \quad 0 \le \varphi \le 2\pi$

$$\begin{pmatrix} x \\ y \end{pmatrix} = (r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} \qquad (\Rightarrow \det(J\Phi(r,\varphi)) = r)$$

2. **Cylindrical Coordinates:**
$0 \le r \le R\,, \quad 0 \le \varphi \le 2\pi\,, \quad a \le z \le b$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (r,\varphi,z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad (\Rightarrow \det(J\Phi(r,\varphi,z)) = r)$$

3. **Spherical Coordinates:**
$0 \le r \le R\,, \quad 0 \le \varphi \le 2\pi\,, \quad -\pi/2 \le \theta \le \pi/2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (r,\varphi,\theta) = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ r\sin\theta \end{pmatrix} \qquad (\Rightarrow \det(J\Phi(r,\varphi,\theta)) = r^2\cos\theta)$$

For continuous functions $f : K \subset \mathbb{R}^n \to \mathbb{R}$, the following holds

$$\int_K f(x)\,dx = \int_D f(\Phi(u)) \cdot |\det(J\Phi(u))|\,du$$

$D \subset \mathbb{R}^n$ compact and measurable, $K = \Phi(D)$, and the C^1 coordinate transformation : $D \to \mathbb{R}^n$.
The transformation Φ must be invertible on D^0.

1. Draw the quarter sphere K given by $y \leq 0$, $z \leq 0$, and $x^2 + y^2 + z^2 \leq 16$. Calculate its center of mass using the density function $\rho(x, y, z) = x^2 + y^2 + z^2 + 1$ and using spherical coordinates.
2. P is described by $x^2 + y^2 + z^2 \leq 9$, there is a sphere K with constant density ρ.
   2.1 Draw K .
   2.2 Calculate the mass and the moment of inertia of K with respect to the z-axis.
   2.3 Calculate the moment of inertia of K with respect to the axis D parallel to the z-axis and passing through the point $(2, 1, 3)^T$.

Example: 04 (1)

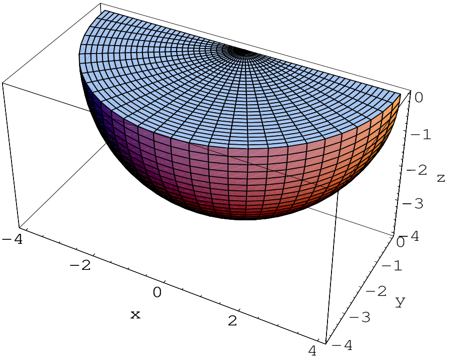

Figure: Quarter-sphere K

Example: 04 (1)

Spherical coordinates for K: $\quad 0 \leq r \leq 4\,,\ \pi \leq \varphi \leq 2\pi\,,\ -\pi/2 \leq \theta \leq 0$
with

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ r\sin(\varphi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix} = \Phi(r,\varphi,\theta), \quad \det J\Phi(r,\varphi,\theta) = r^2\cos(\theta)$$

Calculation of the mass M in spherical coordinates using the transformation theorem with $\rho(x,y,z) = x^2 + y^2 + z^2 + 1$:

$$\begin{aligned} M &= \int_K (x^2 + y^2 + z^2 + 1)\, d(x,y,z) \\ &= \int_0^4 \int_\pi^{2\pi} \int_{-\pi/2}^{0} (r^2+1) r^2 \cos(\theta)\, d\theta\, d\varphi\, dr \\ &= \int_0^4 \int_\pi^{2\pi} r^4 + r^2\, d\varphi\, dr \\ &= \int_0^4 \pi(r^4 + r^2)\, dr = \left.\frac{(3 \cdot r^5 + 5 \cdot r^3)\pi}{15}\right|_0^4 = \frac{3392\pi}{15} \end{aligned}$$

Calculation of the coordinates of the center of mass (x_s, y_s, z_s):

$$
\begin{aligned}
x_s &= \frac{1}{M} \int\limits_K (x^2 + y^2 + z^2 + 1)x \; d(x,y,z) \\
&= \frac{1}{M} \int\limits_0^4 \int\limits_\pi^{2\pi} \int\limits_{-\pi/2}^{0} (r^2+1) r \cos(\varphi)\cos(\theta) r^2 \cos(\theta)\; d\theta\, d\varphi\, dr \\
&= \frac{1}{M} \int\limits_0^4 \int\limits_\pi^{2\pi} (r^5 + r^3)\cos(\varphi) \left. \frac{\theta + \sin(\theta)\cos(\theta)}{2} \right|_{-\pi/2}^{0} d\varphi\, dr \\
&= \frac{\pi}{4M} \int\limits_0^4 (r^5 + r^3)\, \sin(\varphi)|_\pi^{2\pi}\; dr = 0
\end{aligned}
$$

This result also arises due to symmetry.

Example: 04 (1)

$$
\begin{aligned}
y_s &= \frac{1}{M}\int_K (x^2+y^2+z^2+1)y\;d(x,y,z) \\
&= \frac{1}{M}\int_0^4\int_\pi^{2\pi}\int_{-\pi/2}^{0}(r^2+1)r\sin(\varphi)\cos(\theta)r^2\cos(\theta)\;d\theta\;d\varphi\;dr \\
&= \frac{1}{M}\int_0^4\int_\pi^{2\pi}(r^5+r^3)\sin(\varphi)\;\left.\frac{\theta+\sin(\theta)\cos(\theta)}{2}\right|_{-\pi/2}^{0}\;d\varphi\;dr \\
&= \frac{\pi}{4M}\int_0^4(r^5+r^3)\;\cos(\varphi)|_\pi^{2\pi}\;\;dr = \frac{\pi(2\cdot r^6+3\cdot r^4)\big|_0^4}{24M} \\
&= \frac{1120\pi}{3M} = \frac{175}{106}
\end{aligned}
$$

Example: 04 (1)

$$
\begin{aligned}
z_s &= \frac{1}{M}\int_K (x^2+y^2+z^2+1)z \, d(x,y,z) \\
&= \frac{1}{M}\int_0^4 \int_\pi^{2\pi} \int_{-\pi/2}^{0} (r^2+1) r \sin(\theta) r^2 \cos(\theta)\, d\theta \, d\varphi \, dr \\
&= \frac{1}{M}\int_0^4 \int_\pi^{2\pi} (r^5+r^3) \left. \frac{\sin^2(\theta)}{2} \right|_{-\pi/2}^{0} d\varphi \, dr \\
&= -\frac{1}{2M}\int_0^4 (r^5+r^3) \left. \varphi \right|_\pi^{2\pi} \, dr = -\frac{\pi(2\cdot r^6+3\cdot r^4)\big|_0^4}{24M} \\
&= -\frac{1120\pi}{3M} = -\frac{175}{106}
\end{aligned}
$$

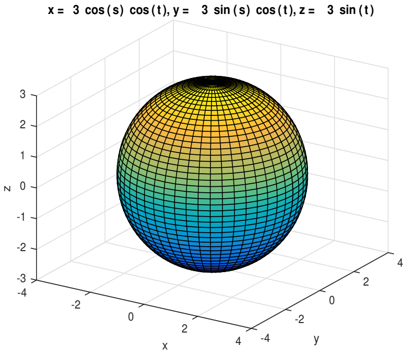

Figure: Sphere K with radius $R = 3$

Calculation of the mass M in spherical coordinates using the transformation theorem with constant density ρ:

$$\begin{aligned} M &= \int\limits_K \rho \, d(x,y,z) = \rho \int\limits_0^3 \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} r^2 \cos(\theta) \, d\theta \, d\varphi \, dr \\ &= \rho \int\limits_0^3 r^2 \, dr \int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} \cos(\theta) \, d\theta \; = \; \rho \left(\frac{r^3}{3} \right) \Big|_0^3 \, (\varphi)|_0^{2\pi} \, (\sin(\theta))|_{-\pi/2}^{\pi/2} \\ &= \rho \frac{3^3}{3} \cdot 2\pi \cdot 2 \; = \; \rho \frac{4\pi 3^3}{3} \; = \; 36\pi\rho \end{aligned}$$

Example: 04 (2.2)

Calculation of the moment of inertia with respect to the z-axis in spherical coordinates using the transformation theorem with constant density ρ and the addition theorem. $\cos^3(\theta) = (3\cos(\theta) + \cos(3\theta))/4$

$$
\begin{aligned}
\Theta_z &= \int_K \rho(x^2+y^2)\, d(x,y,z) \\
&= \rho \int_0^3 \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} (r^2\cos^2(\varphi)\cos^2(\theta) + r^2\sin^2(\varphi)\cos^2(\theta))\, r^2\cos(\theta)\, d\theta\, d\varphi\, dr \\
&= \rho \int_0^3 r^4 dr \int_0^{2\pi} d\varphi \int_{-\pi/2}^{\pi/2} \cos^3(\theta) d\theta \\
&= \rho \left(\frac{r^5}{5}\right)\Bigg|_0^3 (\varphi)|_0^{2\pi}\, \frac{1}{4}\left(3\sin(\theta) + \frac{1}{3}\sin(3\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= \rho\frac{3^5}{5}\cdot 2\pi \cdot \frac{4}{3} \;=\; \frac{648\pi\rho}{5}
\end{aligned}
$$

Example: 04 (2.3)

Since the center of mass of P is at the origin due to symmetry reasons, according to the Steiner's theorem,

$$\begin{aligned}\Theta_D &= Md^2 + \Theta_{z\text{-axis}} \\ &= 36\pi\rho(2^2 + 1^2) + \frac{648\pi\rho}{5} \\ &= \frac{1548\pi\rho}{5}\,.\end{aligned}$$

THANK YOU