Analysis III: Auditorium Exercise-05

For Engineering Students

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December 11, 2023

The solvability of the system of equations is examined

$$\begin{aligned} g_1(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0 \\ &\vdots \\ g_m(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0\,, \end{aligned}$$

briefly denoted as $g(x,y)=0$, for the variable $y\in\mathbb{R}^m$.

In this case, y would be expressible as a function of x,

In the equation $g(x,y)=0$, the function f would be implicitly contained.

Let $g : D \to \mathbb{R}^m$ be a C^1 function defined on the open set $D \subset \mathbb{R}^n \times \mathbb{R}^m$, and consider a point $(x^0, y^0) \in D$ where $x^0 \in \mathbb{R}^n$ and $y^0 \in \mathbb{R}^m$ such that $g(x^0, y^0) = 0$.

Furthermore, assume that the following $m \times m$ submatrix of $Jg(x^0, y^0)$ is regular:

$$\frac{\partial g}{\partial y}(x^0, y^0) := \begin{pmatrix} \frac{\partial g_1}{\partial y_1}(x^0, y^0) & \cdots & \frac{\partial g_1}{\partial y_m}(x^0, y^0) \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1}(x^0, y^0) & \cdots & \frac{\partial g_m}{\partial y_m}(x^0, y^0) \end{pmatrix}$$

Then there exist open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ with $x^0 \in U$, $y^0 \in V$, and $U \times V \subset D$, and a uniquely determined continuously differentiable function

$$f : U \to V$$

such that

$$y^0 = f(x^0) \quad \text{and} \quad g(x, f(x)) = 0 \quad \text{for all} \quad x \in U \ .$$

The Jacobian matrix Jf is computed for all $x \in U$ by differentiating the implicit equation $g(x, f(x)) = 0$ (using the chain rule), which leads to the equation system:

$$\frac{\partial g}{\partial x}(x, f(x)) + \frac{\partial g}{\partial y}(x, f(x)) \cdot Jf(x) = 0.$$

For a C^1-function $g : \mathbb{R}^2 \to \mathbb{R}$, the solution set given by

$$g(x, y) = 0$$

is examined.

The solvability of the equation for one of the variables is guaranteed when $g_x \neq 0$ or $g_y \neq 0$, that is,

$$\operatorname{grad} g = (g_x, g_y) \neq 0$$

The points (x_0, y_0) for which grad $g(x_0, y_0) \neq 0$ are therefore called **regular**.
In regular points, the solution set

$$g = 0$$

is described by a contour line.
In this context, a **horizontal tangent** is present at (x_0, y_0) if

$$g(x_0, y_0) = 0, \quad g_x(x_0, y_0) = 0, \quad g_y(x_0, y_0) \neq 0$$

holds, and a **vertical tangent** for

$$g(x_0, y_0) = 0, \quad g_x(x_0, y_0) \neq 0, \quad g_y(x_0, y_0) = 0.$$

The points (x_0, y_0) for which grad $g(x_0, y_0) = 0$ are called **singular** or **stationary**.

Classification of singular points of $g(x, y) = 0$:

(x_0, y_0) is an **isolated point** if $\det Hg(x_0, y_0) > 0$,

(x_0, y_0) is a **double point** if $\det Hg(x_0, y_0) < 0$.

(x_0, y_0) is a **cusp point** if $\det Hg(x_0, y_0) = 0$.

To investigate the curve implicitly defined by the level set

$$f(x,y) := x^3 + y^3 - xy = 0,$$

we follow the instructions provided.

a) Determine the symmetries of the curve.

The curve is symmetric with respect to the bisector, meaning that $f(x,y) = f(y,x)$. We recall the reflection matrix S_α:

$$\underbrace{\begin{pmatrix} \cos\left(\frac{2\cdot\pi}{4}\right) & \sin\left(\frac{2\cdot\pi}{4}\right) \\ \sin\left(\frac{2\cdot\pi}{4}\right) & -\cos\left(\frac{2\cdot\pi}{4}\right) \end{pmatrix}}_{=S_{\pi/4}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

This reflects the point (x, y) across the line $y = x$.

b) Determine the points on the curve with a horizontal tangent.

$\text{grad} f(x, y) = (3x^2 - y, 3y^2 - x)^T$

Points on the curve with a horizontal tangent are obtained from the conditions

$$f_x(x, y) = 0 \quad \wedge \quad f(x, y) = 0 \quad \wedge \quad f_y(x, y) \neq 0$$

$$0 = f_x(x,y) = 3x^2 - y \quad \Rightarrow \quad y = 3x^2 \quad \Rightarrow$$

$$0 = f(x, 3x^2) = x^3 + (3x^2)^3 - x3x^2 = x^3(27x^3 - 2)$$

$$\Rightarrow \quad x = 0 \quad \vee \quad x = \frac{2^{1/3}}{3}$$

$$\Rightarrow \quad P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; P_1 = \frac{1}{3}\begin{pmatrix} 2^{1/3} \\ 2^{2/3} \end{pmatrix}.$$

Only for P_1 does the condition $f_y(P_1) \neq 0$ hold.

Therefore, P_1 is a point with a horizontal tangent.

c) Determine the points on the curve with a vertical tangent.

Points on the curve with a vertical tangent are obtained from the conditions

$$f_y(x,y)=0 \quad \wedge \quad f(x,y)=0 \quad \wedge \quad f_x(x,y)\neq 0.$$

$$0=f_y(x,y)=3y^2-x \quad \Rightarrow \quad x=3y^2 \quad \Rightarrow$$

$$0=f(3y^2,y)=(3y^2)^3+y^3-3y^2y=y^3(27y^3-2)$$

$$\Rightarrow \quad y=0 \quad \vee \quad y=\frac{2^{1/3}}{3}$$

$$\Rightarrow \quad P_0=\begin{pmatrix}0\\0\end{pmatrix},\ P_2=\frac{1}{3}\begin{pmatrix}2^{2/3}\\2^{1/3}\end{pmatrix}$$

Only for P_2 does the condition $f_x(P_2) \neq 0$ hold.

Therefore, P_2 is a point with a vertical tangent.

This can also be deduced without calculation from the symmetry.

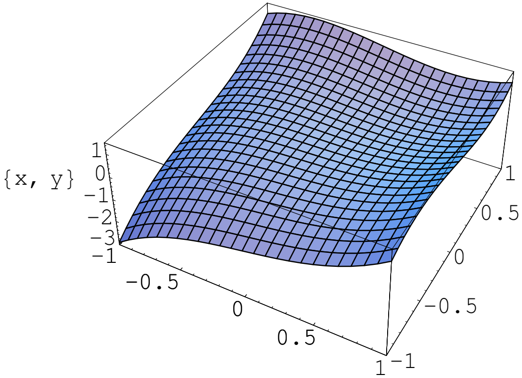
d) Classify the singular points of the curve.

For $P_0 = (0,0)^T$, $\operatorname{grad} f(0,0) = \mathbf{0}$, making P_0 a singular point.

$$Hf(x,y) = \begin{pmatrix} 6x & -1 \\ -1 & 6y \end{pmatrix} \quad \Rightarrow \quad Hf(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Since $\det Hf(0,0) = -1 < 0$, P_0 is a double point.

e) Draw the level set:



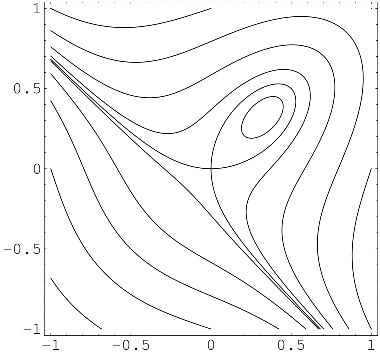


Figure: $f(x,y) = x^3 + y^3 - xy = c$
for $c = -2, -1, -0.5, -0.2, -0.025, 0, 0.05, 0.2, 0.5, 1$

Tangential plane to an implicitly represented surface

The solution set

$$g(x, y, z) = 0$$

of a C^1 function

$$g : \mathbb{R}^3 \to \mathbb{R}$$

describes locally a surface in (x_0, y_0, z_0) with $g(x_0, y_0, z_0) = 0$ if grad $g(x_0, y_0, z_0) \neq \mathbf{0}$.

For example, if $g_z(x_0, y_0, z_0) \neq 0$,
there is solvability with respect to $z = z(x, y)$, with $z_0 = z(x_0, y_0)$.

Tangential plane to an implicitly represented surface

The parametric form of the tangent plane in $\mathbb{R}^3$ to the graph $(x, y, z(x, y))^T$ is then given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z(x_0, y_0) \end{pmatrix} + (x - x_0) \begin{pmatrix} 1 \\ 0 \\ z_x(x_0, y_0) \end{pmatrix} + (y - y_0) \begin{pmatrix} 0 \\ 1 \\ z_y(x_0, y_0) \end{pmatrix}$$

According to the Implicit Function Theorem, we obtain

$$(g_x(x_0, y_0, z_0), g_y(x_0, y_0, z_0)) + g_z(x_0, y_0, z_0)\,(z_x(x_0, y_0), z_y(x_0, y_0)) = (0, 0)$$

and thus

$$(z_x(x_0, y_0), z_y(x_0, y_0)) = -\left(\frac{g_x(x_0, y_0, z_0)}{g_z(x_0, y_0, z_0)}, \frac{g_y(x_0, y_0, z_0)}{g_z(x_0, y_0, z_0)}\right) .$$

The direction vectors of the tangent plane

$$\begin{pmatrix} 1 \\ 0 \\ -\dfrac{g_x(x_0, y_0, z_0)}{g_z(x_0, y_0, z_0)} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -\dfrac{g_y(x_0, y_0, z_0)}{g_z(x_0, y_0, z_0)} \end{pmatrix}$$

are perpendicular to

$$\text{grad } g(x_0, y_0, z_0) = (g_x(x_0, y_0, z_0), g_y(x_0, y_0, z_0), g_z(x_0, y_0, z_0))^T .$$

Example: 02

Given is the function $h : \mathbb{R}^3 \to \mathbb{R}$ with

$$h(x, y, z) = z^2 + y^2 - x^2 + 4z - 2x + 3\,.$$

a) Check whether the level set $h(x, y, z) = c$, determined by the point $(-1, 1, -2)$, forms a smooth surface in the vicinity of this point.

b) Solve the above equation, if necessary, for one of the variables, to explicitly specify the surface.

c) Provide the parametric form of the tangent plane in the point $(-1, 1, -2)$ with respect to the surface from (a).

d) Draw the surface with the tangent plane.

Example: 02 (a)

By completing the square, h can be expressed more clearly:

$$h(x, y, z) = z^2 + y^2 - x^2 + 4z - 2x + 3 = (z+2)^2 + y^2 - (x+1)^2$$

Since $h(-1, 1, -2) = 1$, the level set
turns out to be a single-sheeted hyperboloid and
is thus described by the standardized implicit equation

$$g(x, y, z) := (z+2)^2 + y^2 - (x+1)^2 - 1 = 0$$

To determine whether $g(x, y, z) = 0$
forms a smooth surface in the vicinity of the point $(-1, 1, -2)$,
the conditions of the Implicit Function Theorem must be checked:

$$\operatorname{grad} g(x, y, z) = (-2(x+1), 2y, 2(z+2))^T \Rightarrow$$

$$\operatorname{grad} g(-1, 1, -2) = (0, 2, 0)^T.$$

Thus, only $g_y(-1, 1, -2) = 2$
forms an invertible 1×1 submatrix.

According to the Implicit Function Theorem,
the level set forms a smooth surface,
which can be described by solving $g(x, y, z) = 0$
for y in a neighborhood of $(-1, 1, -2)$,

$$y = f(x, z)\,, \quad \text{with} \quad f(-1, -2) = 1 \quad \text{and} \quad g(x, f(x, z), z) = 0.$$

Solving the implicit equation $g(x, y, z) = 0$ yields initially

$$y = \pm\sqrt{1 + (x+1)^2 - (z+2)^2}.$$

From these two possibilities, it follows,
because $y = f(-1, -2) = 1$

$$f(x, z) = \sqrt{1 + (x+1)^2 - (z+2)^2}\ .$$

Example: 02 (c)

In $(-1, 1, -2)$, the surface f is approximately described by the corresponding tangent plane T_1, in vector-valued notation, this means:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ f(x,z) \\ z \end{pmatrix} \approx \begin{pmatrix} x \\ T_1(x,z;-1,-2) \\ z \end{pmatrix}$$

To represent the tangent plane, the Jacobian matrix of f is required, obtained by implicit differentiation of $g(x, f(x,z), z) = 0$ using the chain rule:

Example: 02 (c)

$$\begin{aligned} \mathbf{J}f(x,z) &= (f_x, f_z) = -(g_y)^{-1}(g_x, g_z) \\ &= -\tfrac{1}{2y}(-2x-2, 2z+4) \\ \Rightarrow \qquad \mathbf{J}f(-1,-2) &= -\tfrac{1}{2\cdot 1}(0,0) = (0,0)\ . \end{aligned}$$

As a reminder:

$$\text{grad } g(x,y,z) = (-2(x+1), 2y, 2(z+2))^T$$

Thus, the parametric form of the tangent plane is

$$\begin{pmatrix} x \\ T_1(x, z; -1, -2) \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x \\ f(-1, -2) + \mathbf{J}f(-1, -2)\begin{pmatrix} x+1 \\ z+2 \end{pmatrix} \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} + (x+1)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (z+2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example: 02 (d)

Using polar coordinates, the surface

$$h(x, y, z) = (z+2)^2 + y^2 - (x+1)^2 = 1$$

can be parameterized as follows for $(r, \varphi) \in [1, R] \times [0, 2\pi]$:

$$y = r\cos\varphi\ ,\ z = r\sin\varphi - 2 \ \Rightarrow\ p_{\pm}(r, \varphi) = \begin{pmatrix} -1 \pm \sqrt{r^2 - 1} \\ r\cos\varphi \\ r\sin\varphi - 2 \end{pmatrix}$$

Example: 02 (d)

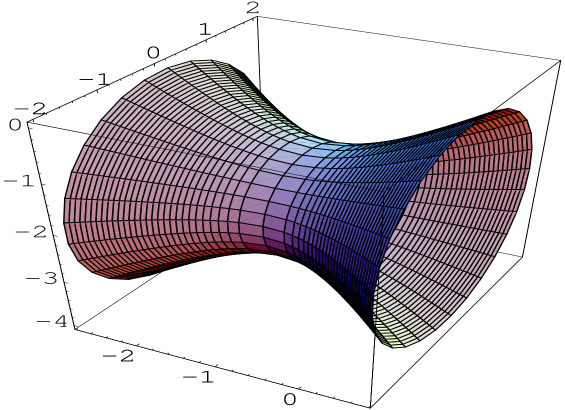


Figure: without tangent plane

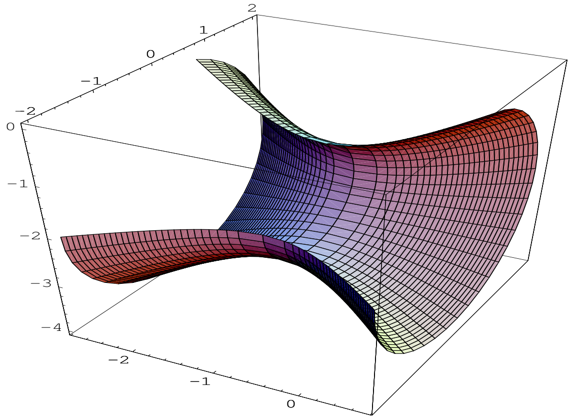


Figure: sliced

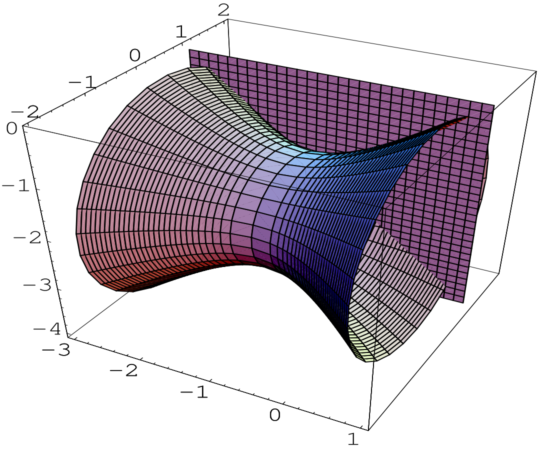


Figure: with tangent plane

The goal is to find the extremal values of a C^1 function

$$f : D \subset \mathbb{R}^n \to \mathbb{R}$$

on the following subset of the domain:

$$G := \{\mathbf{x} \in D \,|\boldsymbol{g}(\mathbf{x}) = \mathbf{0}\} \;\subset\; D,$$

with a C^1 function

$$\boldsymbol{g} : D \to \mathbb{R}^m$$

and $m < n$, i.e., the extremal values must additionally satisfy the m equations

$$\boldsymbol{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))^T = \mathbf{0}$$

Theorem: (Lagrange Multiplier Rule)

Let $\mathbf{x}^0 \in D$ be a local extremum of the function f under the constraint $\boldsymbol{g}(\mathbf{x}^0) = \mathbf{0}$, satisfying the **regularity condition**

$$\text{Rank } \mathbf{J}\boldsymbol{g}(\mathbf{x}^0) = m$$

Then there exist **Lagrange multipliers** $\lambda_1, \ldots, \lambda_m$,
such that the **Lagrange function**

$$F(\mathbf{x}) := f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

satisfies the **necessary first-order condition**:

$$\text{grad } F(\mathbf{x}^0) = \text{grad } f(\mathbf{x}^0) + \sum_{i=1}^{m} \lambda_i \text{grad } g_i(\mathbf{x}^0) = \mathbf{0}.$$

Theorem: (Sufficient Second-Order Condition)

If Rank $\mathbf{J}\boldsymbol{g}(\mathbf{x}^0) = m$ for $\mathbf{x}^0 \in G$
and grad $F(\mathbf{x}^0) = \mathbf{0}$
and $\mathbf{H}F(\mathbf{x}^0)$ is positive definite on

$$TG(\mathbf{x}^0) := \left\{\mathbf{y} \in \mathbb{R}^n \mid \left\langle \text{grad } g_i(\mathbf{x}^0), \mathbf{y} \right\rangle = 0 \right\} \ ,$$

i.e., $\mathbf{y}^T \cdot \mathbf{H}F(\mathbf{x}^0) \cdot \mathbf{y} > 0$ for $\mathbf{y} \in T\mathbf{G}(\mathbf{x}^0) \backslash \{\mathbf{0}\}$,

then f has a strict local minimum at $\mathbf{x}^0$
under the constraint $\boldsymbol{g}(\mathbf{x}) = \mathbf{0}$.

Compute the extremal values of the function

$$f : \mathbb{R}^2 \to \mathbb{R}\,, \quad f(x,y) = x + y$$

on the circle $x^2 + y^2 = 1$.
a) Under the constraint

$$g(x,y) := x^2 + y^2 - 1 = 0$$

determine the extremal points of the function

$$f(x,y) = x + y$$

using the Lagrange multiplier rule.

Example: 03 (a)

Regularity condition:

grad $g(x, y) = (2x, 2y) = (0, 0) \Rightarrow (x, y) = (0, 0)$,

i.e., only $(0, 0)$ violates the regularity condition.

Since $g(0, 0) = -1$, $(0, 0)$ is not on the circle.

All feasible points, i.e., those with $g(x, y) = 0$, satisfy the regularity condition

$$\text{Rank}(\mathbf{J}g(x, y)) = 1.$$

Example: 03 (a)

Lagrangian: $F(x, y) = x + y + \lambda(x^2 + y^2 - 1)$

Lagrange Multiplier Rule:

$$\begin{pmatrix} \nabla F(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} 1 + 2\lambda x \\ 1 + 2\lambda y \\ x^2 + y^2 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Multiplying the first equation by y and the second by x and subtracting both, we get

$$x - y = 0 \quad \Rightarrow \quad x = y.$$

From the third equation, we then obtain $x^2 + x^2 = 1$

$$\Rightarrow \quad x_{1,2} = \pm\frac{1}{\sqrt{2}}\,, \quad y_{1,2} = \pm\frac{1}{\sqrt{2}}.$$

Extremal candidates:

$$P_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad P_2 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since the set $g(x,y)=0$ describes a circle, it is compact.

Thus, the continuous function f attains a maximum and minimum on $g(x,y)=0$.

We have $f(P_1)=\sqrt{2}$ and $f(P_2)=-\sqrt{2}$.

So, P_1 is a maximum and P_2 is a minimum.

Example: 03.(a)

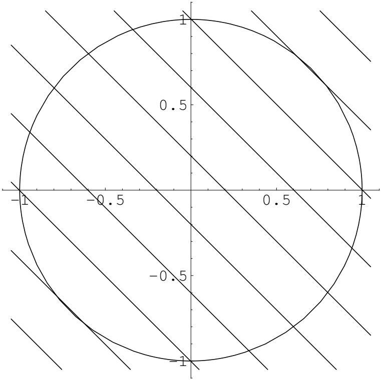


Image: Constraint $g(x, y) = x^2 + y^2 - 1 = 0$
with level curves of the function $f(x, y) = x + y$

Example: 03.(b)

b) Parametrization of the circle

$$g(x, y) := x^2 + y^2 - 1 = 0$$

by c and then solving the extremal problem
for $h(t) := f(c(t))$.

Example: 03.(b)

The circle is parametrized by polar coordinates

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} =: c(t) \,, \quad 0 \le t < 2\pi,$$

i.e., $g(\cos t, \sin t) = 0$.

Now, we just need to find the extrema of the function

$$h(t) := f(c(t)) = \cos t + \sin t$$

$$h'(t) = -\sin t + \cos t = 0 \quad \Rightarrow \quad \tan t = 1$$

$$\Rightarrow \quad t_1 = \frac{\pi}{4} \,, \quad t_2 = \frac{5\pi}{4}$$

Example: 03.(b)

$$h''(t) = -\cos t - \sin t$$

$$\Rightarrow \quad h''(t_1) = -\sqrt{2} < 0\,,\ h''(t_2) = \sqrt{2}$$

Thus,
$t_1 = \pi/4$ is a maximum with $h(t_1) = \sqrt{2}$
and
$t_2 = 5\pi/4$ is a minimum with $h(t_2) = -\sqrt{2}$.

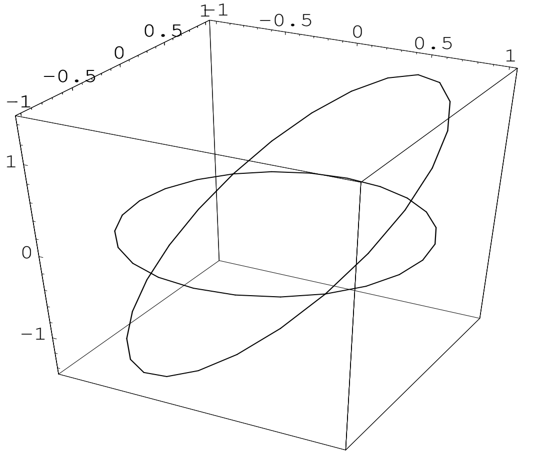


Figure: $c(t)$ and $f(c(t)) = \cos t + \sin t$

For the function

$$f(x, y, z) = z^2$$

compute and classify the extrema on the intersection of the cylinder $x^2 + y^2 = 9$ with the plane $y = z$ using the Lagrange multipliers rule.

Constraints:
$g_1(x, y, z) := x^2 + y^2 - 9$ and $g_2(x, y, z) := y - z$.

Regularization condition:

$$Jg(x, y, z) = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

has rank < 2, when the first row is equal to the zero vector, i.e., for the points $(0, 0, z)$.

However, these are not feasible due to

$$g_1(0, 0, z) = -9$$

So, all feasible points satisfy the regularization condition,
The Lagrange multiplier rule can be applied:

Lagrange function:

$$F(x, y, z) = z^2 + \lambda_1(x^2 + y^2 - 9) + \lambda_2(y - z)$$

Lagrange multiplier rule:

$$\begin{pmatrix} \nabla F(x,y,z) \\ g(x,y,z) \end{pmatrix} = \begin{pmatrix} 2\lambda_1 x \\ 2\lambda_1 y + \lambda_2 \\ 2z - \lambda_2 \\ x^2 + y^2 - 9 \\ y - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

1. Equation:

1. Case: $x = 0$

$\Rightarrow \quad 0 = g_1(0, y, z) = y^2 - 9$

$\Rightarrow \quad y = 3 = z \quad \vee \quad y = -3 = z$

Extreme candidates: $P_1 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}, P_2 = \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}$

2. Case: $\lambda_1 = 0$

$\Rightarrow \quad \lambda_2 = 0 \quad \Rightarrow \quad z = 0 = y \quad \Rightarrow \quad x = 3 \vee x = -3$

Extreme candidates: $P_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, P_4 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$

The intersection of the cylinder $x^2 + y^2 = 9$ with the plane $y = z$ is an ellipse and therefore compact.

The continuous function f attains its absolute maximum and minimum there.

Among the extreme candidates
are the absolute maximum and minimum.

The function values of the extreme candidates are

$$f(P_{1,2}) = 9 \,, \quad f(P_{3,4}) = 0 \,.$$

So, $P_{1,2}$ are absolute maxima, and $P_{3,4}$ are absolute minima.

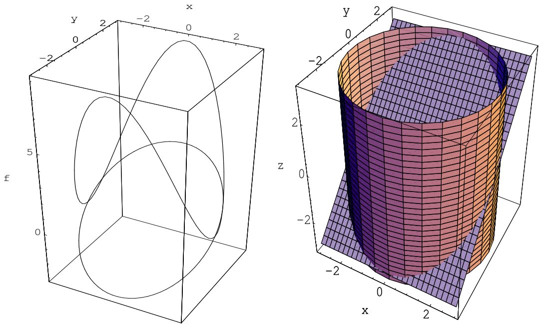


Figure: f on the intersection of the cylinder $x^2 + y^2 = 9$ with the plane $y = z$

THANK YOU