Analysis III: Auditorium Exercise-04

For Engineering Students

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Consider a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ that is m times continuously partially differentiable in D, where D is open and convex, and $n, m \in \mathbb{N}$. Let $x^0 \in D$. Then the Taylor expansion of f at x^0 up to order m is defined as:

$$T_m(x; x^0) := \sum_{j=0}^{m} \frac{1}{j!} \left(\left((x - x^0)^T \nabla \right)^j f \right) (x^0)$$

Alternative representation using **multi-indices**:

α_i Number of derivatives with respect to x_i ,

$\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$

$|\alpha| := \alpha_1 + \cdots + \alpha_n$,

$\alpha! := \alpha_1! \cdot \cdots \cdot \alpha_n!$,

$D^\alpha f := \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$,

$x^\alpha := x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$

$$T_m(x; x^0) = \sum_{|\alpha| \leq m} \frac{D^\alpha f(x^0)}{\alpha!} (x - x^0)^\alpha \,.$$

$$
\begin{aligned}
& T_2(x,y,z;x_0,y_0,z_0) \\
= \quad & f(x_0,y_0,z_0) \\
& +f_x(x_0,y_0,z_0)(x-x_0)+f_y(x_0,y_0,z_0)(y-y_0)+f_z(x_0,y_0,z_0)(z-z_0) \\
& +\frac{1}{2}\left(f_{xx}(x_0,y_0,z_0)(x-x_0)^2+f_{yy}(x_0,y_0,z_0)(y-y_0)^2\right. \\
& +f_{zz}(x_0,y_0,z_0)(z-z_0)^2+2f_{xy}(x_0,y_0,z_0)(x-x_0)(y-y_0) \\
& \left.+2f_{xz}(x_0,y_0,z_0)(x-x_0)(z-z_0)+2f_{yz}(x_0,y_0,z_0)(y-y_0)(z-z_0)\right)
\end{aligned}
$$

$$
\begin{aligned}
& T_3(x, y; x_0, y_0) \\
= \quad & f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
& + \tfrac{1}{2}\big(f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) \\
& \qquad + f_{yy}(x_0, y_0)(y - y_0)^2\big) \\
& + \tfrac{1}{6}\big(f_{xxx}(x_0, y_0)(x - x_0)^3 + 3f_{xxy}(x_0, y_0)(x - x_0)^2(y - y_0) \\
& \qquad + 3f_{xyy}(x_0, y_0)(x - x_0)(y - y_0)^2 + f_{yyy}(x_0, y_0)(y - y_0)^3\big)
\end{aligned}
$$

Calculate the Taylor polynomial of degree 2 for the following function

$$f(x, y, z) = 1 + z + xy + x^2(1 - y)^2 + (y + z)^3$$

around the expansion point $(0, 0, 0)$.

Solution:

$$f(x,y,z) = 1 + z + xy + x^2(1-y)^2 + (y+z)^3 \quad \Rightarrow \quad f(0,0,0) = 1$$

$$\begin{aligned}
f_x(x,y,z) &= y + 2x(1-y)^2 && \Rightarrow \quad f_x(0,0,0) = 0 \\
f_y(x,y,z) &= x - 2x^2(1-y) + 3(y+z)^2 && \Rightarrow \quad f_y(0,0,0) = 0 \\
f_z(x,y,z) &= 1 + 3(y+z)^2 && \Rightarrow \quad f_z(0,0,0) = 1
\end{aligned}$$

$$\begin{aligned}
f_{xx}(x,y,z) &= 2(1-y)^2 && \Rightarrow \quad f_{xx}(0,0,0) = 2 \\
f_{xy}(x,y,z) &= 1 - 4x(1-y) && \Rightarrow \quad f_{xy}(0,0,0) = 1 \\
f_{xz}(x,y,z) &= 0 && \Rightarrow \quad f_{xz}(0,0,0) = 0
\end{aligned}$$

$$\begin{aligned}
f_{yy}(x,y,z) &= 2x^2 + 6(y+z) && \Rightarrow \quad f_{yy}(0,0,0) = 0 \\
f_{yz}(x,y,z) &= 6(y+z) && \Rightarrow \quad f_{yz}(0,0,0) = 0 \\
f_{zz}(x,y,z) &= 6(y+z) && \Rightarrow \quad f_{zz}(0,0,0) = 0
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow T_2(x,y,z;0,0,0) &= f(0,0,0)+f_x(0,0,0)x+f_y(0,0,0)y+f_z(0,0,0)z \\
&+\tfrac{1}{2}\Big(f_{xx}(0,0,0)x^2+f_{yy}(0,0,0)y^2+f_{zz}(0,0,0)z^2 \\
&+2f_{xy}(0,0,0)xy+2f_{xz}(0,0,0)xz+2f_{yz}(0,0,0)yz\Big) \\
&= 1+z+xy+x^2
\end{aligned}
$$

Since the expansion point is the origin, it would have been easier to expand the given function by multiplication and then omit terms beyond the quadratic ones:

$$f(x,y,z)=1+z+xy+x^2-2yx^2+x^2y^2+y^3+3y^2z+3yz^2+z^3\,.$$

Find the 3rd-degree Taylor polynomial of the following function

$$f(x, y) = x \sin(x + y)$$

at the point $\left(0, \frac{\pi}{2}\right)$.

Solution:

$$
\begin{aligned}
f(x,y) &= x\sin(x+y) && \Rightarrow \quad f\left(0,\tfrac{\pi}{2}\right) = 0\\
f_x(x,y) &= \sin(x+y) + x\cos(x+y) && \Rightarrow \quad f_x\left(0,\tfrac{\pi}{2}\right) = 1\\
f_y(x,y) &= x\cos(x+y) && \Rightarrow \quad f_y\left(0,\tfrac{\pi}{2}\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
f_{xx}(x,y) &= 2\cos(x+y) - x\sin(x+y) && \Rightarrow \quad f_{xx}\left(0,\tfrac{\pi}{2}\right) = 0\\
f_{xy}(x,y) &= \cos(x+y) - x\sin(x+y) && \Rightarrow \quad f_{xy}\left(0,\tfrac{\pi}{2}\right) = 0\\
f_{yy}(x,y) &= -x\sin(x+y) && \Rightarrow \quad f_{yy}\left(0,\tfrac{\pi}{2}\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
f_{xxx}(x,y) &= -3\sin(x+y) - x\cos(x+y) && \Rightarrow \quad f_{xxx}(0,\pi/2) = -3\\
f_{xxy}(x,y) &= -2\sin(x+y) - x\cos(x+y) && \Rightarrow \quad f_{xxy}(0,\pi/2) = -2\\
f_{xyy}(x,y) &= -\sin(x+y) - x\cos(x+y) && \Rightarrow \quad f_{xyy}(0,\pi/2) = -1\\
f_{yyy}(x,y) &= -x\cos(x+y) && \Rightarrow \quad f_{yyy}(0,\pi/2) = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow T_3(x,y;0,\pi/2) &= f(0,\pi/2)+f_x(0,\pi/2)x+f_y(0,\pi/2)(y-\pi/2)\\
&\quad +\tfrac{1}{2}\left(f_{xx}(0,\pi/2)x^2+2f_{xy}(0,\pi/2)x(y-\pi/2)\right.\\
&\qquad \left.+f_{yy}(0,\pi/2)(y-\pi/2)^2\right)\\
&\quad +\tfrac{1}{6}\left(f_{xxx}(0,\pi/2)x^3+3f_{xxy}(0,\pi/2)x^2(y-\pi/2)\right.\\
&\qquad +3f_{xyy}(0,\pi/2)x(y-\pi/2)^2+f_{yyy}(0,\pi/2)(y-\pi/2\\
&= x-x^3/2-x^2(y-\pi/2)-x(y-\pi/2)^2/2
\end{aligned}
$$

If f is $(m+1)$ times continuously partially differentiable, then for the **Taylor expansion**

$$f(x) = T_m(x;x^0) + R_m(x;x^0)$$

the following **Lagrange remainder formula** holds,
with $\xi := x^0 + \Theta(x - x^0)$ and $0 < \Theta < 1$

$$R_m(x;x_0) = \frac{1}{(m+1)!}\left(\left((x - x_0)^T\nabla\right)^{(m+1)} f\right)(\xi)$$

Alternatively, in terms of **multi-indices**:

$$R_m(x;x^0) = \sum_{|\alpha|=m+1} \frac{D^\alpha f(\xi)}{\alpha!}(x - x^0)^\alpha \,.$$

Example: 02

$$
\begin{aligned}
R_3(x,y;x_0,y_0) \quad = \quad & +\tfrac{1}{4!}\Big(f_{xxxx}(\xi_1,\xi_2)(x-x_0)^4 \\
& +4f_{xxxy}(\xi_1,\xi_2)(x-x_0)^3(y-y_0) \\
& +6f_{xxyy}(\xi_1,\xi_2)(x-x_0)^2(y-y_0)^2 \\
& +4f_{xyyy}(\xi_1,\xi_2)(x-x_0)(y-y_0)^3 \\
& +f_{yyyy}(\xi_1,\xi_2)(y-y_0)^4\Big)
\end{aligned}
$$

Calculate the 2nd-degree Taylor Polynomial for the point of development $(x_0, y_0) = (0, 0)$ for the following function

$$h(x, y) = \cos(x^2 + y^2)$$

and estimate the error that arises when using T_2 instead of h in the rectangle $[0, \pi/4] \times [0, \pi/4]$, from above.

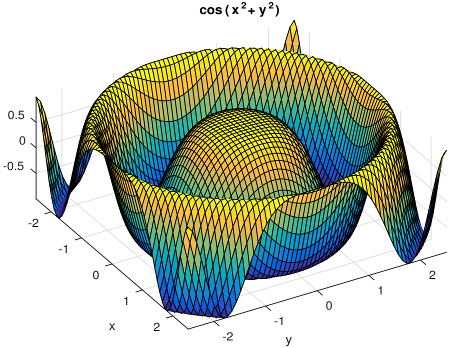
Solution:

$$\begin{aligned}
h(x,y) &= \cos(x^2+y^2) && \Rightarrow \quad h(0,0)=1 \\
h_x(x,y) &= -2x\sin(x^2+y^2) && \Rightarrow \quad h_x(0,0)=0 \\
h_y(x,y) &= -2y\sin(x^2+y^2) && \Rightarrow \quad h_y(0,0)=0 \\
h_{xx}(x,y) &= -2\sin(x^2+y^2)-4x^2\cos(x^2+y^2) && \Rightarrow \quad h_{xx}(0,0)=0 \\
h_{xy}(x,y) &= -4xy\cos(x^2+y^2) && \Rightarrow \quad h_{xy}(0,0)=0 \\
h_{yy}(x,y) &= -2\sin(x^2+y^2)-4y^2\cos(x^2+y^2) && \Rightarrow \quad h_{yy}(0,0)=0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad T_2(x,y;0,0) &= h(0,0)+h_x(0,0)x+h_y(0,0)y \\
&\quad +\tfrac{1}{2}\left(h_{xx}(0,0)x^2+h_{xy}(0,0)xy+h_{yy}(0,0)y^2\right) \\
&= 1
\end{aligned}$$

MATLAB command for surface plot:

ezsurf('cos($x^2 + y^2$)', [−2.5, 2.5, −2.5, 2.5])



For the error estimation, the third derivatives are required:

$$\begin{aligned} h_{xxx}(x,y) &= -12x\cos(x^2+y^2)+8x^3\sin(x^2+y^2) \\ h_{xxy}(x,y) &= -4y\cos(x^2+y^2)+8x^2y\sin(x^2+y^2) \\ h_{xyy}(x,y) &= -4x\cos(x^2+y^2)+8y^2x\sin(x^2+y^2) \\ h_{yyy}(x,y) &= -12y\cos(x^2+y^2)+8y^3\sin(x^2+y^2). \end{aligned}$$

The error estimation for any $(x,y)\in[0,\pi/4]\times[0,\pi/4]$ implies, with $\theta\in]0,1[$, any

$$(\xi_1,\xi_2) := (0,0)+\theta(x,y)\in]0,\pi/4[\times]0,\pi/4[$$

Using the triangle inequality, we obtain:

$$
\begin{aligned}
& |h(x,y) - T_2(x,y;0,0)| = |R_2(x,y;0,0)| \\
= \; & \tfrac{1}{3!}\left|h_{xxx}(\xi_1,\xi_2)x^3 + 3h_{xxy}(\xi_1,\xi_2)x^2y + 3h_{xyy}(\xi_1,\xi_2)xy^2 + h_{yyy}(\xi_1,\xi_2)y^3\right| \\
\leq \; & \tfrac{1}{3!}\left(|h_{xxx}(\xi_1,\xi_2)| \cdot |x|^3 + 3\,|h_{xxy}(\xi_1,\xi_2)| \cdot |x^2y|\right. \\
& \left. +3\,|h_{xyy}(\xi_1,\xi_2)| \cdot |xy^2| + |h_{yyy}(\xi_1,\xi_2)| \cdot |y^3|\right).
\end{aligned}
$$

Each of the four terms can now be individually upper-bounded. Using $|\sin t| \leq 1$ and $|\cos t| \leq 1$, we have:

$$
\begin{aligned}
& |h_{xxx}(\xi_1,\xi_2)| \cdot |x|^3 \\
= \; & \left|-12\xi_1\cos(\xi_1^2+\xi_2^2) + 8\xi_1^3\sin(\xi_1^2+\xi_2^2)\right| \cdot |x|^3 \\
\leq \; & \left(|-12\xi_1| \cdot |\cos(\xi_1^2+\xi_2^2)| + |8\xi_1^3| \cdot |\sin(\xi_1^2+\xi_2^2)|\right) \cdot |x|^3 \\
\leq \; & \left(12 \cdot \frac{\pi}{4} + 8 \cdot \left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3
\end{aligned}
$$

Similarly,

$$3\,|h_{xxy}(\xi_1,\xi_2)|\cdot|x^2y| \le 3\left(4\cdot\frac{\pi}{4}+8\cdot\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$

$$3\,|h_{xyy}(\xi_1,\xi_2)|\cdot|xy^2| \le 3\left(4\cdot\frac{\pi}{4}+8\cdot\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$

$$|h_{yyy}(\xi_1,\xi_2)|\cdot|y^3| \le \left(12\cdot\frac{\pi}{4}+8\cdot\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$ Overall, we have:

$$|h(x,y)-T_2(x,y;0,0)| \le \frac{\pi^3}{3!4^3}\left(48\cdot\frac{\pi}{4}+64\cdot\left(\frac{\pi}{4}\right)^3\right) = 5.5476...$$

The maximum error occurs at $x=y=\frac{\pi}{4}$:

$$|h\left(\frac{\pi}{4},\frac{\pi}{4}\right)-T_2\left(\frac{\pi}{4},\frac{\pi}{4};0,0\right)| = |\cos\left(2\cdot\frac{\pi^2}{4^2}\right)-1| = 0.669252...$$

Let's consider a function f,

$$\begin{array}{rcl} f : D \subset \mathbb{R}^n & \to & \mathbb{R} \\ x & \mapsto & f(x) \end{array}$$

where $x = (x_1, \cdots, x_n)$.

Definition:

For $x^0 \in D$, we define:

- f has a **global maximum** at x^0 if for all $x \in D$, $f(x) \leq f(x^0)$.
- f has a **local maximum** at x^0 if there exists $\varepsilon > 0$ such that for all $x \in D$ with $||x - x^0|| < \varepsilon$, $f(x) \leq f(x^0)$.
- In 1) and 2), if the inequality $f(x) \leq f(x^0)$ can be replaced by $f(x) < f(x^0)$ for $x \neq x^0$, it is a **strict maximum** at x^0.

- If $f(x) \geq f(x^0)$ in 1) and 2), and $f(x) > f(x^0)$ in c), then it is a **minimum** at x^0.
- f has an **extremum** at x^0 if it is either a maximum or minimum.
- f has a **stationary point** at $x^0 \in D$ if $\operatorname{grad} f(x^0) = 0$.

Let f be a C^1 function in D^0, and $x^0 \in D^0$ is a **local extremum**, then

$$\operatorname{grad} f(x^0) = 0.$$

For a twice-partially differentiable function,

$$Hf(x) = \begin{pmatrix} f_{x_1x_1}(x) & \cdots & f_{x_1x_n}(x) \\ \vdots & & \vdots \\ f_{x_nx_1}(x) & \cdots & f_{x_nx_n}(x) \end{pmatrix}$$

represents the **Hessian matrix** of f.

Theorem: Second-Order Necessary Condition

If f is a C^2 function and $x^0 \in D^0$ is a stationary point, then:

1. If $x^0 \in D$ is a **local minimum**,
   then $Hf(x^0)$ is positive semidefinite.

2. If $x^0 \in D$ is a **local maximum**,
   then $Hf(x^0)$ is negative semidefinite.

Theorem: Second-Order Sufficient Condition

If f is a C^2 function and $x^0 \in D^0$ is a stationary point, then:

1. If $Hf(x^0)$ is positive definite,
   then x^0 is a strict local minimum.

2. If $Hf(x^0)$ is negative definite,
   then x^0 is a strict local maximum.

3. If $Hf(x^0)$ is indefinite,
   then x^0 is a **saddle point**.

Example: 02

Compute all stationary points of the following function and classify them:

$$f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$$

Example: 02

grad $f(x, y) = e^{-x^2-y^2}(2x(1-x^2+y^2), 2y(-1-x^2+y^2))^T = (0,0)^T$
To compute the stationary points, we set $f_x(x, y) = 0$ and consider all cases.
Case 1: $x = 0$
$\Rightarrow\ 0 = f_y(0, y) = e^{-y^2}2y(-1+y^2)$
$\Rightarrow\quad y = 0, \quad y = 1, \quad y = -1$
$\Rightarrow$ stationary points:

$$P_1 = (0,0), \quad P_2 = (0,1), \quad P_3 = (0,-1)$$

Case 2: $1 - x^2 + y^2 = 0 \Rightarrow\ x^2 = 1 + y^2$
$\Rightarrow\ 0 = f_y(x, y) = e^{-(1+y^2)-y^2}2y(-1-(1+y^2)+y^2)$
$\quad = -4ye^{-1-2y^2}$
$\Rightarrow\ y = 0 \quad \Rightarrow \quad x = 1, \quad x = -1$
$\Rightarrow$ stationary points: $P_4 = (1,0), \quad P_5 = (-1,0)$

$$Hf(x,y) =$$
$$2e^{-x^2-y^2}\begin{pmatrix} 1-5x^2+2x^4+y^2-2x^2y^2 & 2xy(x^2-y^2) \\ 2xy(x^2-y^2) & -1+5y^2-2y^4-x^2+2x^2y^2 \end{pmatrix}$$

$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ is indefinite

$\Rightarrow \quad P_1 = (0,0)$ is a saddle point.

$Hf(0,\pm 1) = 2e^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is positive definite

$\Rightarrow \; P_{2,3} = (0,\pm 1)$ are minima.

$Hf(\pm 1,0) = -2e^{-1}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is negative definite

$\Rightarrow \; P_{4,5} = (\pm 1,0)$ are maxima.

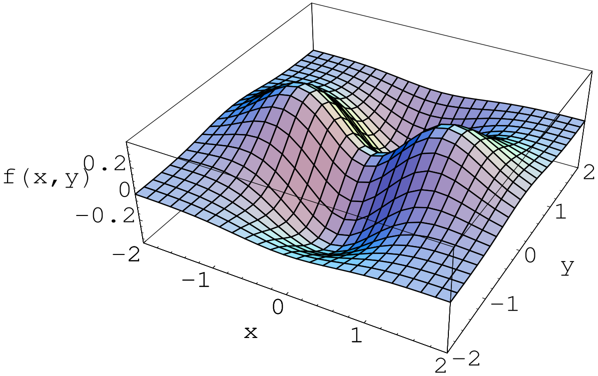


Figure : $f(x,y) = (x^2 - y^2)e^{-x^2-y^2}$

Example: 03

Compute all stationary points of the following function and classify them:

$$f(x, y) = y(y^2 - 3)$$

Example: 03

grad $f(x,y) = (0, 3y^2 - 3)^T = (0,0)^T \ \Rightarrow \ y = \pm 1, \ x \in \mathbb{R}$
The stationary points lie on the lines
$P_1(x) = (x,1)$ and $P_2(x) = (x,-1)$.

$$Hf(x,y) = \begin{pmatrix} 0 & 0 \\ 0 & 6y \end{pmatrix}$$

$Hf(x,1) = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$ is positive semi definite
$\Rightarrow \quad P_1(x) = (x,1)$ are not local maxima.
$Hf(x,-1) = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}$ is negative semidefinite
$\Rightarrow \quad P_2(x) = (x,-1)$ are not local minima.
f is independent of x,
i.e., for fixed $y = c$, $f(x,c) = \textit{constant}$ for all $x \in \mathbb{R}$.

The extrema are thus the ones of $g(y) = y(y^2 - 3)$, i.e., all points on the line $P_1(x) = (x, 1)$ are local minima and for $P_2(x) = (x, -1)$ one obtains local maxima.

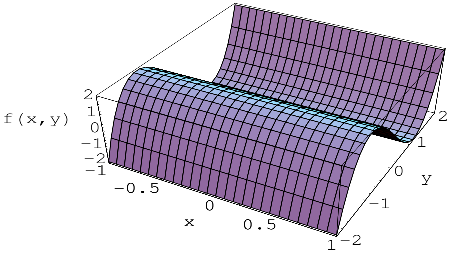


Figure : $f(x, y) = y(y^2 - 3)$

Example: 04

Compute all stationary points of the following function and classify them:

$$f(x, y) = \sin(x^2 + y^2)$$

Solution: grad $f(x, y) = 2\cos(x^2 + y^2)(x, y)^T = (0, 0)^T$

The stationary points are thus given by $(0, 0)$ and all points P, for which $x^2 + y^2 = \pi/2 + n\pi$ with $n \in \mathbb{N}_0$.

$Hf(x, y) =$

$$\begin{pmatrix} 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2) & -4xy\sin(x^2+y^2) \\ -4xy\sin(x^2+y^2) & 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2) \end{pmatrix}$$

$Hf(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is positive definite

$\Rightarrow \quad (0, 0)$ is a minimum.

$Hf(P) = \begin{pmatrix} -4x^2\sin(x^2+y^2) & -4xy\sin(x^2+y^2) \\ -4xy\sin(x^2+y^2) & -4y^2\sin(x^2+y^2) \end{pmatrix}$

is semi definite, as $\det Hf(P) = 0$.

We classify differently:
For points P on the circles $x^2 + y^2 = \pi/2 + n\pi$
we have $\sin(\pi/2 + n\pi) = (-1)^n$.
Therefore, for even n there are maxima, and for odd n there are minima on these circles.

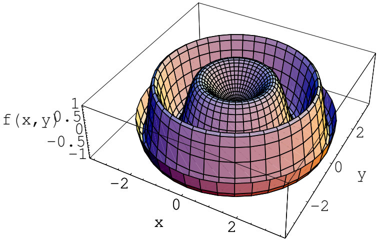


Figure : $f(x, y) = \sin(x^2 + y^2)$

Given the function

$$f(x, y) = 8x^4 - 10x^2y + 3y^2.$$

1. Calculate all stationary points of f
2. Try to apply the sufficient condition for the classification of stationary points.
3. Show that f has a local minimum at the origin along every line through the origin.
4. Does f also have a minimum at the origin along every parabola $y = ax^2$ with $a \in \mathbb{R}$?
5. Plot the function, for example, using the MATLAB routines 'ezsurf' and 'ezcontour'.

Exercise: 04.(1)

Solution:

grad $f(x, y) = (4x(8x^2 - 5y), -10x^2 + 6y)^T = 0$

1. Case: $x = 0$

$\Rightarrow$ $6y = 0$ $\Rightarrow$ stationary point $(x_0, y_0) = (0, 0)$.

2. Case: $8x^2 - 5y = 0$

$\Rightarrow$ $y = 8x^2/5$ $\Rightarrow$ $-10x^2 + 6 \cdot 8x^2/5 = 0$ $\Rightarrow$ $x = 0$ The only stationary point is thus $(0, 0)$.

$$Hf(x,y) = \begin{pmatrix} 96x^2 - 20y & -20x \\ -20x & 6 \end{pmatrix} \quad \Rightarrow \quad Hf(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix}$$ is positive semi definite,

and the sufficient criterion is not applicable.

The necessary condition of order 2 leaves the possibilities of being a minimum or a saddle point for the stationary point

$$(x_0, y_0) = (0, 0).$$

On the line $x = 0$, the function is described by

$$g(y) := f(0, y) = 3y^2.$$

For $y = 0$, g has a strict local minimum. All other origin lines can be represented by $y = ax$ with $a \in \mathbb{R}$ and the function is then described by

$$h(x) := f(x, ax) = 8x^4 - 10ax^3 + 3a^2x^2$$

For $a = 0$, h is minimal at $x = 0$. For $a \neq 0$, a minimum is also obtained at $x = 0$ because

$$h'(x) = 32x^3 - 30ax^2 + 6a^2x \quad \Rightarrow \quad h'(0) = 0$$

and

$$h''(x) = 96x^2 - 60ax + 6a^2 \quad \Rightarrow \quad h''(0) = 6a^2 > 0\,.$$

Exercise: 04.(4)

On the parabola $y = ax^2$, the function takes the form
$p(x) := f(x, ax^2) = 8x^4 - 10ax^4 + 3a^2x^4$
$= x^4(3a^2 - 10a + 8) = x^4(a-2)(3a-4)$. This yields

$$\begin{array}{rclcrcl} p'(x) &=& 4x^3(a-2)(3a-4) &\Rightarrow& p'(0) &=& 0 \\ p''(x) &=& 12x^2(a-2)(3a-4) &\Rightarrow& p''(0) &=& 0 \\ p'''(x) &=& 24x(a-2)(3a-4) &\Rightarrow& p'''(0) &=& 0 \\ p''''(x) &=& 24(a-2)(3a-4) &\Rightarrow& p''''(0) &=& 24(a-2)(3a-4)\,. \end{array}$$

For $a \in\,]4/3, 2[$, $p''''(0) < 0$
and there is a strict maximum at $x = 0$.
For $a \notin [4/3, 2]$, $p''''(0) > 0$
and there is a strict minimum at $x = 0$.

Thus, at the stationary point $(0, 0)$, it is a saddle point. If it were known that

$$f(x, y) = (2y - 3x^2)^2 - (y - x^2)^2$$

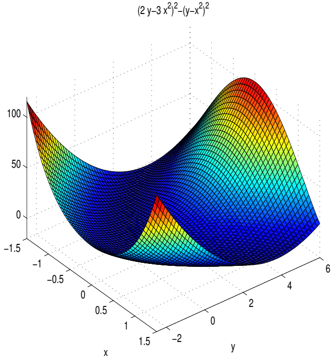
, then on the origin parabola

$$2y - 3x^2 = 0$$

at $x = 0$, an immediate maximum
and on

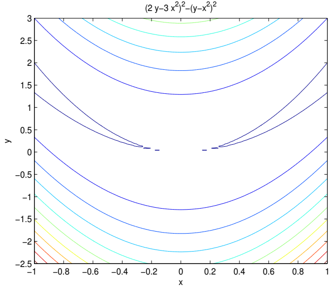
$$y - x^2 = 0$$

at $x = 0$, an immediate minimum would have been recognized and then it would have been immediately inferred to be a saddle point.



ezsurf('8*x$^4 - 10 * x^2 * y + 3 * y^2$', $[-1.5, 1.5, -2.5, 6]$)

Figure: $f(x, y) = 8x^4 - 10x^2y + 3y^2$



ezcontour('8*x$^4 - 10 * x^2 * y + 3 * y^{2\prime}, [-1, 1, -2.5, 3])$

Figure: $f(x, y) = 8x^4 - 10x^2y + 3y^2$

THANK YOU